# *Report 1*

# *Reverberation*

Written by

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2010

**Abstract**

*This document is part of a series of reports covering the topic "Room Acoustics", which was conducted as a self study course with kind assistance from Jonas Brunskog from the Danish Technical University.*

*The objective is the study of the mathematics, starting by solving the wave equation in one and three dimensions. The modes within the room are studied and the effect of wall impedance is analysed. The transfer function is derived and the pulse response becomes an expression with exponentially decaying harmonic oscillations. After this the topic changes to statistic room acoustics covering the development of the equations due to Sabine and Eyring.*

*Basically the report is a reproduction of selected chapters from Kuttruff "Room Acoustics". I needed to acquire the material by repeating some of the derivations and using my own words for the comments. Some examples have been included using a narrow tube to demonstrate the application of the solution within one dimension. The solution to the wave equation within three dimensions uses the rectangular room as a model and the effect of finite wall impedance is studied. The transfer function of the room is derived and transformed into the time domain. The famous Sabine's formula is derived using an assumption of diffuse field, and Eyring's formula is derived using a different approach with the mean free path length between reflections, and I have added a short introduction to the concept of mean free path length using a cube as an example. A couple of derivations were rudimentarily described by Kuttruff so I have added the required equations and comments. I have also added some examples to study selected issues in detail and a couple of sections dealing with the standing waves within the rectangular room.*

# 1. Definitions

***Summary*** – This section presents definitions without proof. This includes the speed of sound in air, the wave equation, equations for deriving the particle velocity and variation in mass density as well as relations for energy density and sound intensity.

## 1.1. Sound

Air is approximated by the *ideal gas law* and the speed of sound becomes as stated below using the static pressure $p_0$ = 100 kPa at sea level and the mass density $\rho_0$ = 1.19 kg/m$^3$ at 20°C. The pressure oscillations is assumed so fast that the associated thermal oscillations cannot level and the process becomes *adiabatic*, which means that relation between pressure $p$ and volume $V$ is given by the relation $pV^\gamma$ = constant, with the parameter $\gamma$ = 1.40 [B-22].

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}} = 343\,\text{m/s}$$

The relation is temperature dependent but this characteristic will be ignored here.

## 1.2. Wave equation

The wave equation is commonly reported as shown below where $\Delta$ is the differential operator according to Laplace, which has different expressions for plane, cylindrical and spherical coordinate systems, as well as more complicated coordinate systems, and $p$ is the sound pressure [K-8].

$$c^2 \Delta p = \frac{\partial^2 p}{\partial t^2}$$

The same equation applies to electromagnetic waves; however, sound waves are longitudinal waves only. The wave equation may be expressed using the particle velocity as a substitute for the sound pressure, or alternatively the mass density or displacement can be applied without changing the form of the solution [B-24]. The equation is also applicable using the temperature variation [K-9]. The boundary conditions must be adjusted to reflect the selected variable.

The representation of $\Delta p$ depends upon the coordinate system being used and is shown below for the rectangular and spherical coordinate systems [RW-250, RW-253]. The rectangular coordinates are useful for the determination of standing waves within a rectangular room or resonances within a tube, but sound waves are anyway propagating as spherical waves. The rather complex expression for spherical waves is in most cases simplified by assuming high degree of symmetry. The spherical representation is only touched lightly within this report, which mainly deals with plane waves.

$$\Delta p_R = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$

$$\Delta p_S = \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\left[\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial p}{\partial \xi} + \frac{1}{1-\xi^2}\frac{\partial^2 p}{\partial \varphi^2}\right], \quad \xi = \cos(\theta)$$

The wave equation can be solved when the boundary conditions are known. For a rectangular room this represents the impedance of the walls of the room, which is commonly assumed infinite to ease the process of determining a solution. However, walls with finite impedance are required for the present work, and Kuttruff gives an introduction to how this problem is attacked.

## 1.3. Harmonic oscillation

In this context only harmonic oscillations will be considered, i.e. oscillations of the form $\cos(\omega t)$ where the angular frequency is $\omega = 2\pi f$, the frequency is $f = 1/T$ and $T$ is the time of one period. The notation according to Euler: $\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t)$ is preferred since the basic function is not changed by differentiation so $N$ times differentiation results in $(i\omega t)^N \exp(i\omega t)$.

$$p = p_0 \exp(i\omega t) \quad \Rightarrow \quad \frac{\partial p}{\partial t} = i\omega p_0 \exp(i\omega t) \quad \Rightarrow \quad \frac{\partial^2 p}{\partial t^2} = (i\omega)^2 p_0 \exp(i\omega t)$$

Insertion of the second-order derivative into the wave equation allow for the elimination of the dependence to time, leaving $\omega$ and $p_0$ representing the harmonic oscillation. The amplitude is still dependent upon position, so the variable is function of the position.

$$c^2\left(\frac{\partial^2 p_0}{\partial x^2} + \frac{\partial^2 p_0}{\partial y^2} + \frac{\partial^2 p_0}{\partial z^2}\right) = -\omega^2 p_0 \quad \text{where} \quad p_0 = p_0(x, y, z)$$

Rearranging this equation leads to the Helmholtz equation, where the indices have been removed since the rest of this report will assume harmonic oscillations so only the amplitude is concerned.

$$\boxed{\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0 \quad \text{where} \quad k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}}$$

## 1.4. Particle velocity

Air particles oscillate around their mean position, thus creating local variations within the sound pressure, but the direction of oscillation is not necessarily the same for all directions. A measure of the directional oscillation is the gradient of the sound pressure, which can be related to the *particle velocity* through an expression known as the *conservation of momentum*. This relates the change in sound pressure in some direction (the gradient of the sound pressure) to the rate of change (the time-derivative) of the particle velocity.

$$\text{grad}(p) = -\rho_0 \frac{\partial \mathbf{v}}{\partial t} \quad \text{where} \quad \text{grad}(p) = \nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$$

For harmonic oscillation, this becomes:

$$\nabla p = -i\omega \rho_0 \mathbf{v}$$

## 1.5. Conservation of mass

Air particles cannot emerge or disappear so the mass of a system must remains constant; hence, for a system that expands, the mass density will decrease to compensate. The expansion of the system is represented by the divergence vector and the rate of change of the density identifies the amount of mass flow.

$$\rho_0 \,\text{div}(\mathbf{v}) = -\frac{\partial \rho(t)}{\partial t} \quad \text{where} \quad \text{div}(\mathbf{v}) = \nabla \bullet p = \frac{\partial \mathbf{v}}{\partial x} + \frac{\partial \mathbf{v}}{\partial y} + \frac{\partial \mathbf{v}}{\partial z}$$

For harmonic oscillation, this becomes:

$$\rho_0 \nabla \bullet \mathbf{v} = -i\omega\rho$$

## 1.6. Resulting sound pressure

Acoustics is generally considered *linear* for sound pressure variations less than the static pressure and particle velocities less than the speed of sound; hence, the principle of superposition applies and the final solution becomes the sum of the partial solutions.

$$p = \sum_n p_n$$

## 1.7. Sound pressure level

The sound pressure level (SPL) is defined as follows, where $p_{RMS}$ is the root mean square of the sound pressure, which is calculated as $p_{RMS} = p/\sqrt{2}$ for sinusoidal oscillations; and $p_{REF}$ represents the threshold of hearing at middle frequencies within the audible range.

$$L = 20\log_{10}\left(\frac{p_{RMS}}{p_{REF}}\right) \quad \text{where} \quad p_{REF} = 20\,\mu\text{Pa}$$

The assumption of sound pressure variations below the static pressure dictates an upper limit far below 191 dB, which is not a problem, at least not for the present work.

> The sound pressure can be measured directly using a conventional condenser microphone, which is available as measuring devices from the Danish companies Brüel & Kjær, GRAS and DPA, or as the cheaper counterpart, the electret-microphones with build-in buffer, available from several manufacturers. The microphones consist of a capacitor where one of the plates is a thin diaphragm, which is free to move as function of the sound pressure. The capacitor is charged to typically 48 V or 200 V from an external source, and the voltage is converted into charge through the equation $Q = CU$, where $Q$ is charge, $C$ is capacitance and $U$ is voltage. The small-signal relation $\Delta U = Q/\Delta C$ is non-linear, but the variation $\Delta C$ is very small and condenser microphones are generally low-distortion devices. The charge of the popular electret-microphone is “frozen” into the device but the build-in impedance buffer requires an external power supply of 5 V at 0.5 mA typical. All these microphones measure the pressure but are insensitive to direction; at least at frequencies well below 20 kHz, and the low-frequency limit is around 1 Hz.

## 1.8. Energy considerations

A vibrating air particle represents energy, which is carried away by the sound wave, and as with any mechanical energy one has to distinguish between potential and kinetic energy density [K-14].

$$w_{pot} = \frac{p^2}{2\rho_0 c^2} \quad \text{and} \quad w_{kin} = \frac{\rho_0 |v|^2}{2} \quad \text{where} \quad w = w_{pot} + w_{kin}$$

In a plane wave the sound pressure and particle velocity are related by $p = \rho_0 c v$, and the same can be assumed for a spherical wave at large distance from the centre, so the particle velocity can be expressed in terms of the sound pressure.

The *energy density* within the plane wave becomes:

$$v = \frac{p}{\rho_0 c} \quad \Rightarrow \quad w = \frac{p^2}{2\rho_0 c^2} + \frac{\rho_0 p^2}{2(\rho_0 c)^2} \quad \Rightarrow \quad w = \frac{p^2}{\rho_0 c^2}$$

The *sound intensity* of the plane wave becomes:

$$I = pv = \frac{p^2}{\rho_0 c} \quad \Rightarrow \quad I = cw$$

# 2. The wave equation

***Summary*** – The wave equation is solved for a couple of simple cases thus generating a basis for the report. Initially one dimension is addressed to introduce the plane wave, and an example is shown for the short tube, which may represent ventilation ducts or musical instruments. The spherical wave is introduced and the acoustic impedance is studied. Finally, the equation is solved within three dimensions and applied to the rectangular room with rigid walls and also to walls with finite impedance. A source is introduced within the room, which leads to the transfer function of the room, and this is transformed to the time domain as an introduction to the exponential decay. The latter part is rudimentary described by Kuttruff so I have added the intermediate material.

## 2.1. Plane wave

The simplest possible case is a system with only one direction. An example is the narrow tube of length $L$. The tube should be so narrow that waves in the other directions cannot exist, which is the case when tube radius $R$ is far less than wavelength. A rule of thumb states that the wavelength should be at least two times larger than the perimeter of the tube [B-29].

$$\lambda = \frac{c}{f} > 4\pi R \quad \Rightarrow \quad f < \frac{c}{4\pi R} \quad \text{or} \quad kR < 0.5$$

A tube of radius $R$ = 0.25 m is thus acoustically narrow up to 110 Hz. The bore of a musical instrument is normally less than 40 mm in diameter so the theory can be applied up to 1.4 kHz in this case.

The Helmholtz equation for propagating plane waves becomes:

$$\frac{\partial^2 p}{\partial x^2} + k^2 p = 0$$

An assumption of a harmonic oscillation with amplitude $p_1$ is easily shown to solve the equation. Using $i^2 = -1$, the second derivative of $p$ becomes $-k^2 p_1$ thus cancelling the term $k^2 p_1$.

$$p = p_1 \exp(ikx) \quad \Rightarrow \quad \frac{\partial p}{\partial x} = p_1 ik \exp(ikx) \quad \Rightarrow \quad \frac{\partial^2 p}{\partial x^2} = p_1 (ik)^2 \exp(kx)$$

More generally, any combination of harmonic oscillations with argument $\pm ikx$ solves the equation, and using the principle of superposition, which is allowed for linear systems, the general solution becomes an infinite sum of harmonic functions. The remaining problem of determining $p^+_n$ and $p^-_n$ as well as $k_n$ is left to the time where the room boundaries are known.

$$p = \sum_n \left( p_n^+ \exp(-ik_n x) + p_n^- \exp(ik_n x) \right)$$

Including the function $\exp(i\omega t)$ representing the oscillation with time, the following expression appears. The argument to the first term is constant for $x = \omega t/k = ct$, which corresponds to a fixed value of the sound pressure moving along the x-axis at the speed of sound. The second term is similarly representing a fixed sound pressure wave moving in the opposite direction. The resulting profile of the sound pressure along the x-axis is thus defined by superposition of wave fronts.

$$\boxed{p = \sum_n \left( p_n^+ \exp(i\omega t - ikx) + p_n^- \exp(i\omega t + ikx) \right)}$$

If $p^-_n = 0$, the solution consists solely of the wave front moving in the positive direction and this can typically be assumed the case for sound sources within a conventional room.

### 2.1.1. Particle velocity

Assuming harmonic oscillation the expression for particle velocity can be related directly to sound pressure. The negative sign indicates that the air particles are flowing in the direction from high pressure toward low pressure and the imaginary unit indicates a phase shift.

$$\frac{\partial v}{\partial t} = i\omega v \quad \Rightarrow \quad \frac{\partial p}{\partial x} = -i\omega\rho_0 v \quad \Rightarrow \quad v = -\frac{1}{i\omega\rho_0}\frac{\partial p}{\partial x}$$

The *particle velocity* of the propagating plane wave can be calculated from the sound pressure at the position of interest, which will be selected to $x = 0$.

$$v = -\frac{1}{i\omega\rho_0}\frac{\partial p}{\partial x}\bigg|_{x=0} = -\frac{1}{i\omega\rho_0}\frac{\partial\{p\exp(-ikx)\}}{\partial x}\bigg|_{x=0} = -\frac{-ikp\exp(-ik0)}{i\omega\rho_0} = \frac{p}{\rho_0 c}$$

The *volume velocity* is the particle velocity integrated through a surface (the unit is $m^3/s$). This is commonly used to describe the source strength of a loudspeaker with the effective area $S$ of the diaphragm. Assuming constant velocity $v$ throughout the diaphragm the quantity is readily found.

$$q = \int_S v dS \quad \Rightarrow \quad q = Sv \quad \text{for } v = \text{constant}$$

### 2.1.2. Acoustical impedance

It is tempting to relate the acoustical parameters to Ohm's law of the electrical circuitry. The sound pressure can be seen as the driven force (analogue to voltage), which is causing the air particles to move (as the electrical current is moving), and the proportionality is the acoustical impedance. The acoustical impedance is thus the sound pressure divided by particle velocity.

The *characteristic impedance* of a propagating wave front is $\rho_0 c$ = 416 kg $m^{-2}$ $s^{-1}$ at 20°C [K-10].

$$p = Zv \quad \Rightarrow \quad Z = \frac{p}{v} = \rho_0 c$$

The *acoustical impedance* is the sound pressure divided by volume velocity.

$$Z_A = \frac{p}{Sv}$$

### 2.1.3. Sound decay

It has silently been assumed that the propagation is within a lossless medium, but this is not the case and sound waves travelling through long paths suffice attenuation due to air absorption. The sound pressure is reduced as shown below [K-110, 161]. The value of $m$ is frequency dependent with three values shown at 20°C, 100 kPa and 60 % relative humidity [K-163].

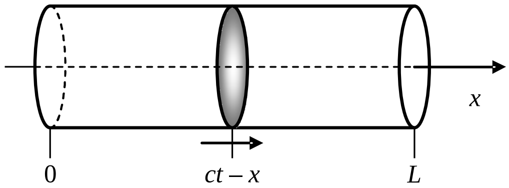
$$p = \sum_n \left( p_n^+ \exp(i\omega t - kx) \exp\left(-\frac{mx}{2}\right)\right) \quad \text{where} \quad \begin{array}{lcccl} m = & 0.64 & 2.14 & 20.5 & \text{m}^{-1} \\ f = & 0.5 & 2 & 8 & \text{kHz} \end{array}$$

Although not proportional to frequency the ratio $m_1 = m/f$ may serve as a useful approximation with the geometrical mean value $m_1 = 1.5 \cdot 10^{-6}$ $m^{-1}$ $Hz^{-1}$. Distance is proportional to time $x = ct$ so the damping factor becomes 0.3 $s^{-1}$ at 1 kHz, which is below the typical range of 1 to 20 $s^{-1}$ for rooms [K-83], thus indicating that the dominating factor for a typical room is surface absorption.

$$\exp\left(-\tfrac{1}{2}mx\right) = \exp\left(-\tfrac{1}{2}m_1 fct\right) = \exp(-\delta t) \quad \Rightarrow \quad \delta = \tfrac{1}{2}m_1 fc = 0.28\,\text{s}^{-1} \quad f = 1\,\text{kHz}$$

### 2.1.4. Example with a short tube

The solution to the wave equation with one variable will be used to solve the problem of standing waves within a narrow tube of length $L$, where the end terminations can be closed or open.




**A narrow tube of length *L* enables only one dimension for wave propagation. The gray disk at the centre represents a section of a plane wave propagating in the positive direction through the tube at the speed of sound. At the end of the tube the wave is reflected thus changing the direction. If the tube is driven at a single frequency many such planes will be moving in both directions and the superposition of the waves generates points with high sound pressure amplitude and other points with low sound pressure amplitude; this is known as standing waves.**

*Both ends closed*

The air particles cannot move along the x-axis at the closed end located at $x = 0$, so the air velocity is zero at this point. A similar argument applies to the end at $x = L$. This corresponds to infinite impedance at both ends of the tube, so high sound pressure is possible even with negligible particle velocity according to the equation $p = Zv$ with $Z \to \infty$. The particle velocity is shown in section 1.4 and leads to the boundary conditions for the narrow tube with blocked ends.

$$\frac{\partial p}{\partial x} = -i\omega\rho_0 v\Big|_{x=0 \text{ or } x=L} = 0 \quad \Rightarrow \quad \frac{\partial p(0)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p(L)}{\partial x} = 0$$

The expression for the general solution is differentiated and equated to zero for $x = 0$, which is the first boundary condition, showing that the amplitude of the two propagating wave fronts are equal.

$$\frac{\partial p(0)}{\partial x} = \frac{\partial}{\partial x}\left\{ p_n^+ \exp(-ik_n x) + p_n^- \exp(ik_n x)\right\} \quad \Rightarrow$$
$$\frac{\partial p(0)}{\partial x} = -p_n^+ k_n \exp(-ik_n x) + p_n^- k_n \exp(ik_n x) \quad \Rightarrow$$
$$-p_n^+ ik_n + p_n^- ik_n = 0 \quad \Rightarrow \quad p_n^+ = p_n^-$$

This requirement is inserted into the equation and the derivative at $x = L$ is equated to the zero, which is the second boundary condition. The result is a trigonometric equation.

$$\frac{\partial p(L)}{\partial x} = -p_n^+ ik_n \exp(-ik_n L) + p_n^+ ik_n \exp(ik_n L) = 0 \quad \Rightarrow$$
$$-p_n^+ ik_n[\cos(k_n L) - i\sin(k_n L)] + p_n^+ ik_n[\cos(k_n L) + i\sin(k_n L)] = 0 \quad \Rightarrow \quad -2p_n^+ k_n \sin(k_n L) = 0$$

The requirement that $\sin(k_n L)$ must be zero is satisfied for $k_n L$ equal to an integer multiply of $\pi$. Hence, the tube will resonate at the frequency $c/2$L and harmonics to this frequency.

$$k_n = \frac{n\pi}{L} \quad \Rightarrow \quad f_n = \frac{nc}{2L}, \quad n = 1, 2, \ldots$$

The tube oscillates at $c/2L$ with both odd and even harmonics. For $L = 1$ m the lowest frequency is at $f_1 = 172$ Hz and the next higher frequency becomes $f_2 = 343$ Hz. This may represent the situation within a room at low frequency if the room is driven in some way.

### *One end closed and one end open*

The end at $x = 0$ is still blocked but the end at $x = L$ is open to radiate into the free space outside the tube. The air particles are allowed to move freely so there is no pressure build-up. The impedance is negligible at the open end, so the sound pressure is low even at high particle velocity according to the equation $p = Zv$ (since $Z \approx 0$); this is called a *pressure relief*. The boundary conditions become:

$$\frac{\partial p(0)}{\partial x} = 0 \quad \text{and} \quad p(L) = 0$$

The expression at $x = L$ leads to the requirement that $\cos(k_n L)$ must be zero, which is satisfied for $k_n L$ equal to $\pi/2$ plus an integer multiply of $\pi$.

$$p(L) = p_n^+ \exp(-ik_n L) + p_n^+ \exp(ik_n L) = 0 \quad \Rightarrow$$
$$p_n^+[\cos(k_n L) - i\sin(k_n L)] + p_n^+[\cos(k_n L) + i\sin(k_n L)] = 0 \quad \Rightarrow \quad 2 p_n^+ k_n \cos(k_n L) = 0$$

The requirement for resonance within the tube becomes:

$$k_n L = \frac{\pi}{2} + n\pi = \frac{(1+2n)\pi}{2} \quad \Rightarrow \quad f_n = \frac{(1+2n)c}{4L}, \quad n = 1, 2, \ldots$$

### *Both ends open*

The air particles are allowed to move freely at $x = 0$ and $x = L$ so the boundary conditions become:

$$p(0) = 0 \quad \text{and} \quad p(L) = 0$$

The first requirement shows that the coefficients must be identical with exception of sign inversion and the second requirement leads to a sine function, which must produce zero.

$$\left\{ \begin{array}{c} p(0) = p_n^+ + p_n^+ = 0 \quad \Rightarrow \quad p_n^+ = -p \\ p(L) = p_n^+ \exp(-ik_n L) - p_n^- \exp(ik_n L) = 0 \end{array} \right\} \quad \Rightarrow$$
$$p_n^+[\cos(k_n L) - i\sin(k_n L)] - p_n^+[\cos(k_n L) + i\sin(k_n L)] = 0 \quad \Rightarrow \quad -2 p_n^+ k_n \sin(k_n L) = 0$$

The requirement for resonance within the tube becomes:

$$k_n L = n\pi \quad \Rightarrow \quad k_n = \frac{n\pi}{L} \quad \Rightarrow \quad f_n = \frac{nc}{2L}, \quad n = 0, 1, 2, \ldots$$

#### ***Musical instruments***

The tube with one end closed and one end open resonates at $c/4L$ and odd harmonics. This represents the design of the musical instrument the *Clarinet* as well as the *Stopped Diapason* organ pipe. For $L = 1$ m the lowest harmonic frequency is $f_1 = 86$ Hz and the next higher frequency is $f_2 = 257$ Hz, which is one and a half octave above the fundamental. For the Clarinet the consequence is that the *overblown* notes are one fifth higher than the octave thus complicating fingering on the instrument. The Stopped Diapason is characterised by missing even harmonics due to the lack of resonance at these frequencies. The organ pipe is driven from the open end using an oscillating air stream (flue). The resonator of the Clarinet is driven through the closed end through a reed, which is closed for most of the time and opens shortly to enter a short burst of air at the frequency of interest.

The tube with both ends open oscillates at $c/2L$ and even harmonics. For $L = 1$ m the lowest frequency is at $f_1 = 172$ Hz and the next higher frequency becomes $f_2 = 343$ Hz. This represents a renaissance *Recorder* or the modern *Flute* as well as the *Open Diapason* organ pipe. The Recorder and Flute is overblown into the octave (double frequency) and the Open Diapason can amplify all harmonics through the resonances (at least for the lower 10 harmonics or so where the theory fits with reality). The double-open tube can be overblown to the octave thus keeping the fingering (almost) fixed for the higher octave of the instrument.

## 2.2. Spherical waves

The expression of the Laplace operator for the spherical coordinate system is not attractive, but may be simplified using the symmetry to remove the terms related to the angles. The consequence is that the wave form is assumed to propagate freely in any direction along radius. Without the dependency to either of the angles, the derivatives with respect to angle become zero and the cumbersome terms disappear. The spherical form of the wave equation is then reduced to two terms.

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

Assuming harmonic oscillation the right-hand side becomes $-k^2 p$, and the wave equation simplifies to a differential equation with the sound pressure being dependent solely upon radius.

$$\frac{\partial^2 p}{\partial t^2} = \frac{(i\omega)^2 p}{c^2} = -k^2 p \quad \Rightarrow \quad \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} + k^2 p = 0 \quad \text{where} \quad p = p(r)$$

Multiplying with $r^2$ brings the equation to a form, which is identified as a special case of the Bessel equation where the parameter $n$ is zero [RW-277].

$$r^2 \frac{\partial^2 p}{\partial r^2} + 2r\frac{\partial p}{\partial r} + k^2 r^2 p = 0 \quad \Rightarrow \quad r^2 p'' + 2rp' + \left(k^2 r^2 - n(n+1)\right)p = 0$$

The solution is given through the spherical Bessel functions [RW-277]:

$$p = Aj_n(kx) + By_n(kx) \quad \text{where} \quad j_0 = \frac{\sin(kx)}{kx} \quad \text{and} \quad y_0 = -\frac{\cos(kx)}{kx}$$

The complete solution becomes:

$$p = a\frac{\sin(kr)}{kr} + b\frac{\cos(kr)}{kr}$$

Using the Euler formulas [RW-62] this is transformed into the usual form with the sum of a wave propagating in the positive direction (away from the source) and another wave propagating in the negative direction (toward the source). A wave front approaching the centre is generally considered unrealistic [K-12], although theoretically possible for a sound field within a sphere with rigid shell.

$$\begin{aligned}\cos(kx) &= \frac{\exp(ikr) + \exp(-ikr)}{2} \\ \sin(kx) &= \frac{\exp(ikr) - \exp(-ikr)}{2i}\end{aligned} \quad \Rightarrow \quad p = p_r^+ \frac{\exp(-ikr)}{kr} + p_r^- \frac{\exp(ikr)}{kr} \quad \text{where} \quad \begin{aligned} p_r^+ &= \frac{b + ia}{2} \\ p_r^- &= \frac{b - ia}{2}\end{aligned}$$

As a conclusion, spherical propagation away from the source is described in an analogues form to the plane wave, although with a decaying term related to distance.

$$\boxed{p = p_r \frac{\exp(-ikr)}{kr}}$$

This is the *distance law* of real acoustic fields. The use of wave number $k$ in the denominator is due to the arrangement of the differential equation prior to solving and is not a requirement for the spherical wave front and any unwanted relation to frequency can be compensated by adjustments of the factor $p_r$. Hence, the equation is often written as: $p = A \exp(-ikr)/r$.

### 2.2.1. Particle velocity

The particle velocity will be expressed using the sound pressure of the outward propagating wave through differentiation of the expression for $p$, using the rule $D(f/g) = (f'g - fg')/g^2$ [RW-137] and the definition $k = \omega/c$. The expression is function of distance and frequency [B-36].

$$v = -\frac{1}{i\omega\rho_0}\frac{\partial}{\partial r}\left\{\frac{p_r^+ \exp(-ikr)}{kr}\right\}\Bigg|_{r=0} = -\frac{p_r^+}{i\omega\rho_0}\frac{-ik\exp(-ikr)kr - \exp(-ikr)k}{(kr)^2} = \frac{p}{\rho_0 c}\frac{1+ikr}{ikr}$$

At low frequencies or close to the source distance ($kr \to 0$) the particle velocity is 90° behind the sound pressure and the amplitude of the particle velocity decays with $r^{-2}$, while the sound pressure decays with $r^{-1}$. At high frequencies or far from the source ($kr \to \infty$) the particle velocity is in phase with the sound pressure and the particle velocity decays with the same rate.

$$v \underset{kr\to 0}{\longrightarrow} \frac{1}{ikr}\frac{p}{\rho_0 c} \quad \text{versus} \quad v \underset{kr\to\infty}{\longrightarrow} \frac{p}{\rho_0 c}$$

### 2.2.2. Acoustical impedance

The characteristic impedance of the spherical wave front is function of both frequency and distance but the value approaches the characteristic impedance of air at $kr \to \infty$. A practical use of this fact is that the spherical wave can be regarded as *locally plane,* thus allowing the theory for plane waves to be used to limited areas of a spherical wave. This represents the *far field* distance from the source and will be assumed to apply within this report.

$$Z = \frac{p}{v} = \rho_0 c\frac{ikr}{1+ikr} \quad \Rightarrow \quad Z \underset{kr\to\infty}{\longrightarrow} \rho_0 c$$

At $kr = 1$ the characteristic impedance is $\sqrt{2}$ lower than the asymptotical value and the phase is lagging the sound pressure by 45°. For low frequencies or short distance ($kr < 1$) the approximation is proportional to both frequency and distance; this is the *near field*.

$$Z \underset{kr\to 0}{\longrightarrow} i\omega\rho_0 r$$

## 2.3. Rectangular room

A rectangular room will be analysed with dimensions of $L_1$, $L_2$ and $L_3$, which may represents length, width and height of the room. The room is assumed to be placed with one corner at the origin of the coordinate system. The Helmholtz equation can be solved by assuming that the function for the sound pressure can be separated into the three directions as shown below.

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0 \quad \text{where} \quad p = p_x p_y p_z$$

The solution is expected to generate an infinite series of partial solutions, shown below for the x-axis, where $C_n$ is a constant to be determined using the boundary conditions and $p_n$ is the $n$'th partial solution. The series decays for increasing $n$ thus gradually approaching the final solution.

$$p_x = \sum_{n_x} C_n p_n$$

The boundary condition form the room is that the particle velocity must be zero at the boundary since the particles cannot move through the walls. This is equivalent of stating that the gradient of the sound pressure must be zero at the boundary, and it is shown below for the x-coordinate and the two other coordinates are defined correspondingly.

$$\frac{\partial p(0)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p(L_x)}{\partial x} = 0$$

The function for the sound pressure is differentiated with respect to the three dimensions. Since $p_x$ is constant to variation within the yz-coordinates the differentiation is straightforward with only $p_x$ being differentiated while $p_y$ and $p_z$ are unaffected, and similar for the other coordinates.

$$\frac{\partial p}{\partial x} = p''_x p_y p_z \quad \text{and} \quad \frac{\partial p}{\partial y} = p_x p''_y p_z \quad \text{and} \quad \frac{\partial p}{\partial z} = p_x p_y p''_z$$

Insertion yields the equation to solve.

$$p''_x p_y p_z + p_x p''_y p_z + p_x p_y p''_z + k^2 p = 0$$

The equation is divided by the sound pressure and the common terms are removed. Division by zero is avoided since $p = 0$ is a trivial solution, which does not give any information about the system. The resulting equation consists of three terms. The first term depends solely upon $x$, the second term depends solely upon $y$ and the third term depends solely upon $z$. This sum must equate a constant, which is possible only if the individual terms are constant. The constant can be written as consisting of individual constants for each dimension.

$$\frac{p''_x}{p_x} + \frac{p''_y}{p_y} + \frac{p''_z}{p_z} = -k^2 \quad \Rightarrow \quad k^2 = k_x^2 + k_y^2 + k_z^2$$

The equations are then solved individually with the solution along the x-axis shown below; and this is an ordinary homogenous differential equation with an analytical solution [K-73].

$$\frac{p''_x}{p_x} = -k_x^2 \quad \Rightarrow \quad p''_x + k_x^2 p_x = 0 \quad \Rightarrow \quad p_x = a_x \cos(k_x x) + b_x \sin(k_x x)$$

The constant $b_x$ is determined using the first boundary conditions. The derivative is zero at the origin ($x = 0$), which requires that the coefficient to the second term is zero.

$$\frac{\partial p_x(0)}{\partial x} = 0 \quad \Rightarrow \quad -a_x k_x \sin(k_x 0) + b_x k_x \cos(k_x 0) = 0 \quad \Rightarrow \quad b_x = 0$$

At the other end ($x = L$) the same requirement must be met by the remaining part of the solution, which is $p_x = a_x \cos(k_x x)$. This leads to the requirement $\sin(k_x L_x) = 0$ so the argument $k_x L_x$ must equal an integral multiple of π. The useful values of $k_x$ forms an infinite series of discrete values.

$$\frac{dp_x(L_x)}{\partial x} = 0 \quad \Rightarrow \quad -a_x k_x \sin(k_x L_x) = 0 \quad \Rightarrow \quad k_k = \frac{n_x \pi}{L_x} \quad \Rightarrow \quad f_x = \frac{n_x c}{2L_x}$$

Constant $a_x$ corresponds to the sound pressure and is defined from the boundary conditions, and the value is non-zero due to the initial requirement of a non-trivial solution. The other two coordinates are determined similarly and the general solution is the product of the solutions.

$$p_n = C_n \cos\left(\frac{n_x \pi x}{L_x}\right)\cos\left(\frac{n_y \pi y}{L_y}\right)\cos\left(\frac{n_z \pi z}{L_z}\right) \quad n = f(n_x, n_y, n_z)$$

The constants $a_x$, $a_y$ and $a_z$ are assembled into the constant $C_n$ representing the sound pressure at point $(x,y,z)$ within the room, and the equation must be completed by multiplication with $\exp(i\omega t)$ describing the time dependence. Index $n$ represents the combination of $n_x$, $n_y$ and $n_z$ and the method used for the examples within this report is defining an upper limit $N$ of each parameter and using this as radix as shown below. However, the series of data was rearranged after the calculation so an increasing index corresponds to an increasing frequency.

$$n = n_x + N n_y + N^2 n_z \quad \text{where} \quad n_x, n_y, n_z = 0, 1, \ldots, N-1, \quad N = 2, 3, \ldots$$

The resultant sound pressure becomes the sum of the particular solutions to the wave equation.

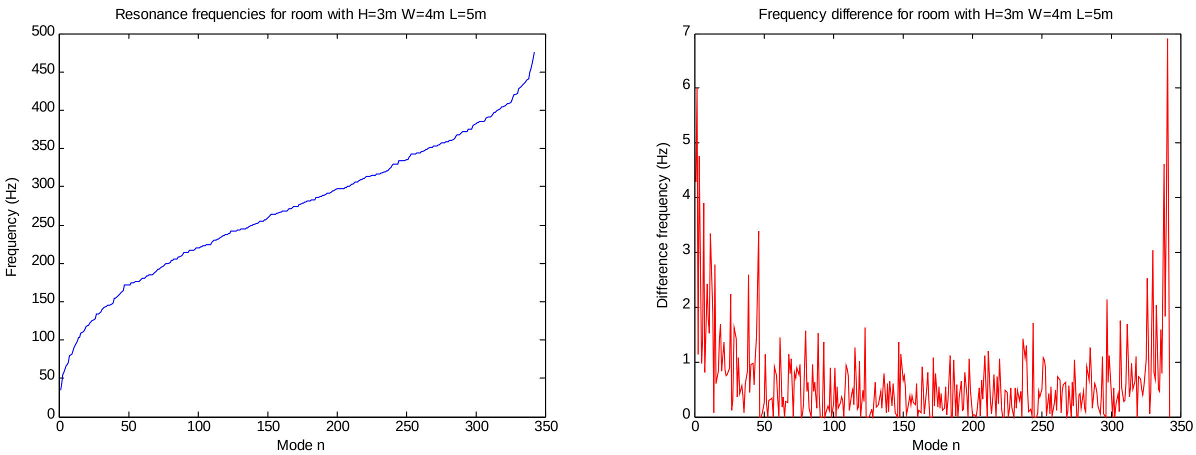
$$p = \sum_n C_n p_n$$

The sound pressure is at its maximum near the surface where the cosine is unity so excitation of the room modes should locate the source (loudspeaker) close to the edge between walls, and preferably close to one of the corners where the orthogonal planes are all available. The sound pressure is zero at all positions when one or more of the cosines are zero; the *nodal planes*.

The frequency of each of the standing waves is calculated from the definition $k^2 = k_x^2 + k_y^2 + k_z^2$ by using $k = 2\pi f/c$. The result is an extension of the plane wave within a narrow tube (section 2.1.4).

$$f_n = \frac{c}{2}\sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2}$$

The node frequencies (or resonance frequencies) are shown below for a rectangular room together with the difference between the neighbouring nodes, which shows that although the frequencies are not evenly spaced, the distance between adjacent frequencies is generally less than 3 Hz.




**Plot of node frequencies for a rectangular room (left) and the difference between neighbouring nodes (right) for a room with the measures: $L_x$ = 3 m, $L_y$ = 4 m and $L_z$ = 5 m. The lowest resonance**

**frequencies are 57 Hz along the x-axis, 43 Hz along the y-axis, and 34 Hz along the z-axis. The average value of the difference frequencies was 0.6 Hz with *N* = 7 and is reduced at higher values.**

All possible modes within the rectangular room are covered by the expression, and not just waves bouncing back and forth between two parallel walls. The resonance modes are one-dimensional (parallel to either of the axes) if only one parameter is different from zero; two-dimensional (*tangential modes*) if two parameters are non-zero; and three-dimensional (*oblique modes*) if all parameters are different from zero.

### 2.3.1. Upper frequency limit

The theory cannot be assumed to apply at elevated frequencies where the wavelength becomes comparable to the depth of windows cavities and other irregularities, so this issue will be lightly investigated (this is not part of the book by Kuttruff). For a wavelength of at least 10 times the deepth $d$ of the irregularity, the upper limit becomes $f_{MAX}$ = 340 Hz at $d$ = 0.1 m.

$$\lambda = \frac{c}{f_{MAX}} = 10d \quad \Rightarrow \quad f_{MAX} = \frac{c}{10d}$$

The useful range of the modes is thus limited and it will be shown that the volume $V$ and surface area $S$ of the room can be used to determine the useful range of the modes, thus disregarding the actual shape of the room.

At the highest frequency we have $n_x = n_y = n_z = N - 1$.

$$f_{MAX} = \frac{c}{2}\sqrt{\left(\frac{N-1}{L_x}\right)^2 + \left(\frac{N-1}{L_y}\right)^2 + \left(\frac{N-1}{L_x}\right)^2} = \frac{c(N-1)}{2}\sqrt{\frac{1}{L_x^2} + \frac{1}{L_y^2} + \frac{1}{L_z^2}}$$

The terms are combined and the volume of the room is identified as $V = L_x L_y L_x$.

$$f_{MAX} = \frac{c(N-1)}{2}\sqrt{\frac{L_x^2 L_y^2 + L_x^2 L_z^2 + L_y^2 L_z^2}{L_x^2 L_y^2 L_z^2}} = \frac{c(N-1)}{2V}\sqrt{L_x^2 L_y^2 + L_x^2 L_z^2 + L_y^2 L_z^2}$$

The remaining terms within the square root represents the square of the area of three of the surfaces. The square of the surface area consists of six terms, including the three terms above.

$$S = 2(L_x L_y + L_x L_z + L_y L_z)$$
$$S^2 = 4(L_x^2 L_y^2 + L_x^2 L_z^2 + L_y^2 L_z^2 + 2L_x^2 L_y L_z + 2L_x L_y L_z^2 + 2L_x L_y^2 L_z)$$

Assuming that $L_x < L_y < L_z$ we may substitute $L_x^2 L_y L_z$ with $L_x^2 L_y^2$ thus reducing the sum ($L_z \to L_y$), substitute $L_x L_y L_z^2$ with $L_x^2 L_z^2$ thus reducing the sum ($L_y \to L_x$) and substitute $L_x L_y^2 L_z$ with $L_y^2 L_z^2$ thus increasing the sum ($L_x \to L_z$), and also noting that the corrections tends to cancel since the increment is larger than each of the two reductions. The approximate value of the surface area is inserted into the equation for the maximum frequency.

$$S^2 \approx 12(L_x^2 L_y^2 + L_x^2 L_z^2 + L_y^2 L_z^2) \quad \Rightarrow \quad f_{MAX} = \frac{c(N-1)}{2V}\sqrt{\frac{S^2}{12}}$$

The highest useful mode number is now determined using the expression of the maximum useful mode frequency $f_{MAX}$, and the value is solely determined by the room constants ($V$ and $S$) and the irregularities of the room ($d$).

$$f_{MAX} = \frac{c}{10d} = \frac{c(N-1)S}{2V\sqrt{12}} \quad \Rightarrow \quad N-1 = \frac{2\sqrt{12}}{10}\frac{V}{Sd} \quad \Rightarrow \quad N \approx 1 + 0.7\frac{V}{Sd}$$

For a room with measure 3 m by 4 m by 5 m the volume is 60 $m^2$, the surface area is 94 $m^2$ and assuming an irregularity of 0.1 m the highest useful mode number becomes $N \approx 5.5$, so modes above $N = 6$ should be avoided – at least as a guideline. For a large room with measure 10 m by 30 m by 50 m the volume is 15000 $m^3$, the surface area is 4600 $m^2$ and assuming an irregularity of 0.25 m the highest useful mode number becomes $N \approx 10$. The lowest resonance frequency is 3.4 Hz and the highest useful frequency becomes $f_{MAX}$ = 137 Hz. Modes will exist at higher frequencies but the above equation cannot be trusted.

### 2.3.2. Distribution of modes

The number of modes within one, two and three dimensions is generated through the parameter *N*, but there is not a constant relation between the modes; the three-order modes dominate for a typical room. The ratio between modes was found[1] through analysing all combinations of $n_x$, $n_y$ and $n_z$ with fixed upper limit of $N - 1$. The equations below use $N_1$ for first-order modes, $N_2$ for second-order modes and $N_3$ for third-order modes and state the number of modes for given value of *N*.

$$N_1 = 3(N-1), \quad N_2 = 3(N-1)^2, \quad N_3 = (N-1)^3$$

The second-order modes outnumbers the first-order modes at $N > 2$ and the third-order modes outnumbers the lower modes for $N > 4$. The ratio between the modes is shown.

$$\frac{N_2}{N_1} = N-1 \quad \text{and} \quad \frac{N_3}{N_2} = \frac{N-1}{3} \quad \Rightarrow \quad (N_1 : N_2 : N_3) = \left(1 : N-1 : \frac{(N-1)^2}{3}\right)$$

Limiting the number of useful modes to $N = 7$, as a compromise based upon the comments from the previous section, produce the relation 1:6:12 between the number of modes with one, two and three dimensions. This result is used to introduce the derivation of Eyring's formula (section 3.2.1).

### 2.3.3. Mode-related absorption

For walls with different absorption coefficients the attenuation of a specific mode is related to how often the mode reaches the wall. For absorbing layers at the walls at $x = 0$ and $x = L_x$ the modes along the x-axis are attenuated for every reflection, while the modes along the y-axis and the z-axis are virtually unaffected, so the first-order modes show different delay rates with this arrangement of the absorption material. The two-dimensional modes will be affected for two-third of the possible reflections, since the modes are located within the planes along *xy*, *xz* and *yz*, but all third-order modes will be affected with one-third of the reflections since they use all six surfaces.

### 2.3.4. Finite wall impedance

The surface of the room is not perfectly rigid in most situations, which means that the particle velocity may have non-vanishing values along the boundary. The analysis becomes involved for a rigorous treatment, so the present analysis will be limited to nearly rigid walls.

The particle velocity can be expressed as function of the sound pressure through the introduction of the impedance $Z_W$ of the wall [K79].

---

[1] This was derived by counting the number of modes with none, one or two zeros for *N* up to 10 (see the MATLAB code at the end of the document). The values were $(N_1, N_2, N_3)$ = (3, 3, 1) for $N = 2$, (6, 12, 8) for $N = 3$, (9, 27, 27) for $N = 4$ and (12, 48, 64) for $N = 5$.

$$v_x = -\frac{1}{ik\rho_0 c}\frac{\partial p_x}{\partial x} \quad \text{and} \quad v_x = \frac{p_x}{Z_W} \quad \Rightarrow \quad \frac{\partial p_x}{\partial x} = -ik\rho_0 c\frac{p_x}{Z_W}$$

The particle velocity at $x = 0$ is positive for direction against the wall, i.e. in the negative direction of the x-axis, which corresponds to a sign inversion, while the particle velocity at $x = L$ is positive for direction along the x-axis. Hence, the boundary conditions along the x-axis where $k$ represents the frequency through $k = \omega/c$.

$$\frac{\partial p_x(0)}{\partial x} = \frac{ik\rho_0 c}{Z_W}p_x(0) \quad \text{and} \quad \frac{\partial p_x(L_x)}{\partial x} = -\frac{ik\rho_0 c}{Z_W}p_x(L_x)$$

The general expression for the sound pressure is reproduced below along with the derivative with respect to distance. The useful values of $k_x$ are to be determined, but for a start the procedure targets the system of equations in the variables $p_x^+$ and $p_x^-$.

$$p_x = p_x^+\exp(-ik_x x) + p_x^-\exp(ik_x x) \quad \Rightarrow$$
$$\frac{\partial p_x(x)}{\partial x} = -ik_x p_x^+\exp(-ik_x x) + ik_x p_x^-\exp(ik_x x)$$

The expressions are inserted into the boundary condition for $x = 0$.

$$-ik_x p_x^+ + ik_x p_x^- = \frac{ik\rho_0 c}{Z_W}\left(p_x^+ + p_x^-\right) \quad \Rightarrow \quad \left(\frac{ik\rho_0 c}{Z_W} + ik_x\right)p_x^+ + \left(\frac{ik\rho_0 c}{Z_W} - ik_x\right)p_x^- = 0$$

The expressions are inserted into the boundary condition for $x = L_x$.

$$-ik_x p_x^+\exp(-ik_x L_x) + ik_x p_x^-\exp(ik_x L_x) = -\frac{ik\rho_0 c}{Z_W}p_x^+\exp(-ik_x L_x) - \frac{ik\rho_0 c}{Z_W}p_x^-\exp(ik_x L_x) \quad \Rightarrow$$
$$\left(-\frac{ik\rho_0 c}{Z_W} + ik_x\right)p_x^+\exp(-ik_x L_x) + \left(-\frac{ik\rho_0 c}{Z_W} - ik_x\right)p_x^-\exp(ik_x L_x) = 0$$

The solution to the set of equations is infinite in the variables $p_x^+$ and $p_x^-$ if the determinant is zero; hence, this requirement is used to guarantee non-trivial solutions for $k_x$.

$$\det\begin{pmatrix} p_x^+ \\ p_x^- \end{pmatrix} = \left(\frac{ik\rho_0 c}{Z_W} + ik_x\right)\left(-\frac{ik\rho_0 c}{Z_W} - ik_x\right)\exp(ik_x L_x) - \left(\frac{ik\rho_0 c}{Z_W} - ik_x\right)\left(-\frac{ik\rho_0 c}{Z_W} + ik_x\right)\exp(-ik_x L_x) = 0$$

This produces the following equation for the determination of $k_x$:

$$\left(-\left(\frac{ik\rho_0 c}{Z_W}\right)^2 - 2ik_x\frac{ik\rho_0 c}{Z_W} - (ik_x)^2\right)\exp(ik_x L_x) = \left(-\left(\frac{ik\rho_0 c}{Z_W}\right)^2 + 2ik_x\frac{ik\rho_0 c}{Z_W} - (ik_x)^2\right)\exp(-ik_x L_x)$$

The terms are assembled and are then multiplied with the square of the impedance and are divided by the square of the characteristic impedance of air in nominator and denominator. Also $i^2 = -1$ is used to remove most of the negative signs.

$$\exp(2ik_xL_x) = \frac{-\left(\frac{ik\rho_0 c}{Z_W}\right)^2 + 2ik_x\frac{ik\rho_0 c}{Z_W} - (ik_x)^2}{-\left(\frac{ik\rho_0 c}{Z_W}\right)^2 - 2ik_x\frac{ik\rho_0 c}{Z_W} - (ik_x)^2} = \frac{k^2 - 2kk_x\frac{Z_W}{\rho_0 c} + \left(k_x\frac{Z_W}{\rho_0 c}\right)^2}{k^2 + 2kk_x\frac{Z_W}{\rho_0 c} + \left(k_x\frac{Z_W}{\rho_0 c}\right)^2}$$

The fractions are recognised as the binomials $(x - y)^2 = x^2 - 2xy + y^2$ for the nominator and similarly $(x + y)^2 = x^2 + 2xy + y^2$ for the denominator, which enables the removal of the square root although the sign becomes unknown; however, this allows for a change of the sequence of terms.

$$\exp(2ik_xL_x) = \frac{\left(k - k_x\frac{Z_W}{\rho_0 c}\right)^2}{\left(k + k_x\frac{Z_W}{\rho_0 c}\right)^2} \quad \Rightarrow \quad \exp(ik_xL_x) = \pm\frac{k - k_x\frac{Z_W}{\rho_0 c}}{k + k_x\frac{Z_W}{\rho_0 c}} = \pm\frac{k_x\frac{Z_W}{\rho_0 c} - k}{k_x\frac{Z_W}{\rho_0 c} + k}$$

The resulting transcendental equation cannot be solved analytically [K-80] but an approximation to the solution is possible through assuming that $k$ is far less than $k_xZ_W/\rho c$ and using the Taylor series expansion of the exponential: $\exp(x) = 1 + x + x^2/2 + \ldots$ with the series terminated after the second term [RW-196]. This results in the following approximate identity[2]:

$$\exp(ik_xL_x) = \pm\left(1 - \frac{\rho_0 ck}{k_xZ_W}\right)$$

The right-hand side expression is now recognised as an exponential and the arguments to the two exponentials must be equal with exception of the uncertainty of $2\pi$ associated with trigonometric functions. This can be accomplished by multiplying the right-hand side by $\exp(in_x\pi)$, which at the same time includes both signs.

$$\exp(ik_xL_x) = \exp\left(in_x\pi - \frac{\rho_0 ck}{k_xZ_W}\right) \quad \Rightarrow \quad ik_xL_x = in_x\pi - \frac{\rho_0 ck}{k_xZ_W} \quad \text{where} \quad n_x = 1, 2, \ldots$$

The last term is small according to the assumption on nearly rigid walls so an intermediate value becomes $k_x = n_x\pi/L_x$ and after re-insertion into the last term, we get an approximate result where the real part is identical to the solution with perfectly rigid walls, so the imaginary part corresponds to the impedance of the nearly rigid wall [K-81]:

$$\boxed{k_x \approx \frac{n_x\pi}{L_x} + \frac{i\rho_0 ck}{n_x\pi Z_W}}$$

For a *mass-controlled* wall the impedance is purely imaginary and proportional to frequency as shown below, where $a$ is a parameter related to the design of the wall. Insertion produces a real value of the correction and the correction term decreases at increasing frequency since the wall impedance is proportional to frequency and the situation approaches that of rigid walls.

$$Z_W = i\rho_0 cka \quad \Rightarrow \quad k_x \approx \frac{n_x\pi}{L_x} + \frac{i\rho_0 ck}{n_x\pi i\rho_0 cka} = \frac{n_x\pi}{L_x} + \frac{1}{n_x\pi a}$$

---

[2] Kuttruff arrives at the expression $1 - 2\rho k/k_xZ_W$ within the parenthesis, but he does not state why the second term must be multiplied by two.

For *resistive losses* in addition to the mass control the impedance includes a real part $R_A$ (with the unit: kg m$^{-2}$ s$^{-1}$). Assuming that $R_A << \rho_0 cka$, we get the following approximate expression where $R_A$ and $a$ are known when the parameters of the wall have been determined. The first two parts of $k_x$ represents the angular frequency of the resonance and approximates $f_n \approx n_x c/2L_x$ since $1/n_x \pi a$ is insignificant, while the imaginary part represents the damping coefficient (see section 2.3.6).

$$Z_W = R_A + i\rho_0 cka \quad \Rightarrow \quad k_x \approx \frac{n_x \pi}{L_x} + \frac{i\rho_0 ck}{n_x \pi (R_A + i\rho_0 cka)} = \frac{n_x \pi}{L_x} + \frac{i\rho_0 ck(R - i\rho_0 cka)}{n_x \pi \left(R_A^2 + (\rho_0 cka)^2\right)}$$

$$k_x \approx \frac{n_x \pi}{L_x} + \frac{1}{n_x \pi a} + \frac{iR_A}{n_x \pi \rho_0 cka}$$

The examination of nearly rigid walls will not be brought further. The conclusion is that all modes are associated with some damping so oscillations within the room decay toward zero due to the dissipation of energy when the sound wave is reflected from the wall.

### 2.3.5. Transfer function

Until now the room has been free of sources, so the theory represents the standing waves, which could exist within the room; however, nothing is oscillating. These so-called *eigenvalues* merely represents frequencies at which the room is capable of sustaining a resonance. Excitation of the room assume sources distributed across the room according to a density function $q(\mathbf{r})$ where $\mathbf{r}$ is a vector, and it is assumed that all sources are oscillating in phase at the same frequency. The density function has the dimension s$^{-1}$ so the volume velocity becomes $q(\mathbf{r})dV$ for the volume element $dV$ at the position $\mathbf{r}$ within the room. The source adds matter to the system so the mass density of the system is changed with the new term $\rho_0 q(\mathbf{r})$. Assuming harmonic oscillation the time derivative is changed into $i\omega$ times the mass density to which the contribution from the source is added [K-69].

$$\rho_0 \nabla \bullet v = -\frac{\partial \rho}{\partial t} + \rho_0 q(\mathbf{r}) = -i\omega\rho + \rho_0 q(\mathbf{r})$$

The gradient of the sound pressure is related to the time derivative of the particle velocity, which is written as $i\omega v$ assuming harmonic oscillations. The divergence operator is applied to both sides and the divergence of the gradient is recognised as the Laplacian of the sound pressure.

$$\nabla p = -\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -i\omega\rho_0 \mathbf{v} \quad \Rightarrow \quad \nabla \bullet \nabla p = -i\omega\rho_0 \nabla \bullet \mathbf{v} = (i\omega)^2 \rho - i\omega\rho_0 q(\mathbf{r})$$

The angular wave number $k = \omega/c$ is introduced.

$$\Delta p + \omega^2 \rho = -i\omega\rho_0 q(\mathbf{r}) \quad \Rightarrow \quad \Delta p + k^2 c^2 \rho = -i\omega\rho_0 q(\mathbf{r})$$

Sound pressure is[3] $p = c^2\rho$, so the wave equation with distributed source becomes [K-69]:

$$\Delta p + k^2 p = -i\omega\rho_0 q(\mathbf{r})$$

The source density can be described through the use of the orthogonal set of functions, which is already derived for the rectangular room, thus re-using the $p_n$ functions although with change of coefficients. The density function becomes expressed as a series expansion.

[3] The translation from mass density to sound pressure was not announced by Kuttruff but his solution induces the use of the relation.

$$q(\mathbf{r}) = \sum_n C_n p_n(\mathbf{r})$$

Determination of the coefficient $C_n$ uses the assumption of orthogonal functions, so the volume integral of the product of $p_n$ and $p_r$ is zero except for the situation where indices are equal ($n = r$). Applying this to the product of $p_n$ and $q$ yields an expression with $C_n$ and $p_n$ squared.

$$\int_V p_n(\mathbf{r}) q(\mathbf{r}) dV = \int_V p_n(\mathbf{r}) \sum_r C_r p_r(\mathbf{r}) dV = \sum_r \int_V p_n(\mathbf{r}) C_r p_r(\mathbf{r}) dV = \int_V C_n p_n^2(\mathbf{r}) dV$$

The coefficient $C_n$ is thus determined when the functions for $p_n(\mathbf{r})$ and $q(\mathbf{r})$ are known. The volume integral of $p_n$ squared is represented through the coefficient $K_n$ to simplify writing [K-69].

$$C_n = \frac{\int_V p_n(\mathbf{r}) q(\mathbf{r}) dV}{\int_V p_n^2(\mathbf{r}) dV} = \frac{1}{K_n} \int_V p_n(\mathbf{r}) q(\mathbf{r}) dV$$

The resulting sound pressure consists of an infinite sum of terms each given by a coefficient $D_n$ and the particular solution $p_n$ and the objective at this point is the determination of the coefficient.

$$p_\omega(\mathbf{r}) = \sum_n D_n p_n(\mathbf{r})$$

Insertion of the definitions into the wave equation creates a system of equations.

$$\sum_n D_n \left( \Delta p_n(\mathbf{r}) + k^2 p_n(\mathbf{r}) \right) = -i\omega\rho_0 \sum_n C_n p_n(\mathbf{r})$$

The addition of sources does not change the arrangement of standing waves within the room, so the particular solution to the homogeneous equation $\Delta p_n + k^2 p_n = 0$ applies and is inserted into the sum.

$$\Delta p_n = -k_n^2 p_n \quad \Rightarrow \quad \sum_n D_n \left( -k_n^2 p_n(\mathbf{r}) + k^2 p_n(\mathbf{r}) \right) = -i\omega\rho_0 \sum_n C_n p_n(\mathbf{r})$$

The common terms are removed and each term in the left-hand sum must equate the corresponding term in the right-hand sum. The equation for determination of $D_n$ is thus found [K-70].

$$D_n = \frac{i\omega\rho_0}{k_n^2 - k^2} C_n$$

Now assume that the density function is concentrated into one point $\mathbf{r}_0$. This point source can be represented by the source $Q$ (with dimension $s^{-1}$) and the Dirac pulse [K-70].

$$q(\mathbf{r}) = Q\delta(\mathbf{r} - \mathbf{r}_0)$$

The coefficient $C_n$ is determined from the previous definition by integration, and since the Dirac pulse is zero at all points within the room except at $\mathbf{r}_0$ the coefficient is found.

$$C_n = \frac{1}{K_n} \int_V p_n(\mathbf{r}) q(\mathbf{r}_0) dV = \frac{1}{K_n} \int_V p_n(\mathbf{r}) Q\delta(\mathbf{r} - \mathbf{r}_0) d\mathbf{r}_0 = \frac{Q}{K_n} p_n(\mathbf{r}_0)$$

Insertion into the expression for $D_n$ leads to an expression with a difference within the denominator, which becomes close to zero when the driven frequency (represented through $k$) approaches the $n$'th normal mode of the room. This means that the coefficient becomes large for frequencies close to the resonance frequency, while the value is limited for frequencies at some distance from the mode; so the $n$'th mode dominates the sum of terms for frequencies close to the frequency of resonance.

$$D_n = \frac{i\omega\rho_0}{k_n^2 - k} \frac{Q}{K_n} \, p_n(\mathbf{r}_0)$$

The sound pressure is finally determined through insertion into the expression for $p_\omega$.

$$p_\omega(\mathbf{r}) = \sum_n D_n p_n(\mathbf{r}) = \sum_n \frac{i\omega\rho_0}{k_n^2 - k^2} \frac{Q}{K_n} \, p_n(\mathbf{r}_0)\, p_n(\mathbf{r})$$

Rearrangement and using $k = \omega/c$ leads into the resulting expression of the sound pressure at the observation point $\mathbf{r}$ due to a source located at position $\mathbf{r}_0$, with the source strength $Q$, the frequency through $k$, the room parameter $k_n$ and an additional parameter $K_n$.

$$\boxed{p_\omega(\mathbf{r}) = ik\rho_0 cQ \sum_n \frac{p_n(\mathbf{r})\, p_n(\mathbf{r}_0)}{\left(k_n^2 - k^2\right) K_n}}$$

This is known as Green's expression and the symmetry allows interchanging of $p_n(\mathbf{r})$ and $p_n(\mathbf{r}_0)$. The point source (the loudspeaker) and the monitoring point (the microphone) can thus be swapped without changing the measured frequency response.

***Reciprocity principle***

This is an important relation within the field of acoustics allowing a complicated setup to be changed into a simpler one. An example could be the measurement of the effect of a loudspeaker baffle using a point source. A point source removes the directional characteristic of the loudspeaker from the measurement, thus leaving the effect of the baffle to be determined, but this requires access to a point source. Through interchanging loudspeaker and microphone, the small mechanical dimensions of a microphone may serve as a fair approximation of a point and the loudspeaker is now placed in the far field where the size can be neglected. This is known as the reciprocity principle.

Green's expression will be used to derive an equation for the transfer function of the room.

### *2.3.6. Decaying modes*

The complex eigenvalue can be written as consisting of an angular frequency $\omega_n$ and damping constant $\delta_n$, with the latter within the range from 1 to 20 $s^{-1}$ according to Kuttruff and the damping constant will be assumed insignificant in comparison to the eigenfrequency [K-83].

$$k_n = \frac{\omega_n}{c} + i\,\frac{\delta_n}{c}$$

This is inserted into Green's expression.

$$p_\omega(\mathbf{r}) = ik\rho_0 cQ \sum_n \frac{p_n(\mathbf{r})\, p_n(\mathbf{r}_0)}{\left(\left(\frac{\omega_n}{c} + i\,\frac{\delta_n}{c}\right)^2 - k^2\right) K_n}$$

After multiplication in nominator and denominator with $c^2$, substituting $c^2k^2 = \omega^2$ and ignoring the squared damping constant $\delta_n^2$ since $\delta_n << \omega_n$ was assumed, we arrive at the transfer function of the room. The parameters describing the room and the location of source and monitor are assembled into the parameter $A_n$.

$$p_\omega(\mathbf{r}) = \sum_n \frac{A_n}{\omega_n{}^2 - \omega^2 + i2\omega_n\delta_n} \quad \text{where} \quad A_n = \frac{i\omega\rho_0 c^2 Q}{K_n} p_n(\mathbf{r}) p_n(\mathbf{r}_0)$$

Each individual term within the sum "blows up" for frequencies at or close to the characteristic frequency where $\omega^2 - \omega_n^2$ becomes less than $2i\delta\omega_n$ as was stated before for the parameter $D_n$. For frequencies distant from the resonance, the equation approaches zero. The room consists of numerous resonances so the resulting transfer function from the source to the listener is build from a superposition of many resonant circuits. Since the term $i2\omega_n\delta_n$ is active only for values close to $\omega_n$ and the bandwidth is relatively narrow, it is allowable to substitute $\omega_n$ by $\omega$ without introducing any serious error [K-83].

$$p_\omega(\mathbf{r}) = \sum_n \frac{A_n}{\omega_n{}^2 - \omega^2 + i2\omega\delta_n}$$

The impulse response of the room is determined by exciting the room with a very short pulse at time $t = 0$. Since the room is at rest for $t < 0$, the transient response is obtained from the Laplace transform with the variable $\omega$ substituted by $s = \alpha + i\omega$, where $\alpha$ represents damping and $\omega$ is the angular frequency[4]. Using $i^2 = -1$ the negative sign to $\omega$ is removed by the substitution. The resultant equation within the denominator is identified as the product of two first-order terms with parameters $a$ and $b$ defined by the damping constant $\delta_n$ and the angular resonance frequency $\omega_n$.

$$p_s = \sum_n \frac{A_n}{s^2 + 2\delta s + \omega_n^2} = \sum_n \frac{A_n}{(s+a)(s+b)} \quad \text{where} \quad \begin{aligned} a &= \delta_n + i\omega_n \\ b &= \delta_n - i\omega_n \end{aligned}$$

Within the Laplace transform (as well as within other integral transforms) the output response to an input signal is the Laplace transform of the input signal (the Dirac pulse) multiplied by the Laplace transform of the impulse response of the circuit. The Laplace transform of the Dirac pulse is $s$, and the circuit has already been transformed, so the transfer function becomes as shown below. The solution within the time domain is obtained from a table of Laplace transform pairs [RW-332].

$$p_s = \sum_n A_n \frac{s}{(s+a)(s+b)} \quad \leftrightarrow \quad p(t) = A_n \frac{a\exp(-at) - b\exp(-bt)}{a-b}$$

Inserting the values of $a$ and $b$ leads to an expression with complex exponentials.

$$p(t) = \sum_n A_n \frac{-\delta_n(\exp(i\omega_n t) - \exp(-i\omega_n t)) + i\omega_n(\exp(i\omega_n t) + \exp(-i\omega_n t))}{i2\omega_n} \exp(-\delta_n t)$$

Using the Euler equations [RW-62] the expressions are substituted by cosine and sine.

$$p(t) = \sum_n A_n \left( \cos(\omega_n t) - \frac{\delta_n}{\omega_n} \sin(\omega_n t) \right) \exp(-\delta_n t)$$

Using the assumption of $\delta_n << \omega_n$ simplifies the expression into a sum of damped oscillations with the frequencies and damping constants defined by the eigenvalue $k_n$.

$$\boxed{p(t) = \sum_n A_n \cos(\omega_n t) \exp(-\delta_n t)}$$

[4] Kuttruff refers to the Fourier transform, without stating details of the derivation, and his solution includes an arbitrary phase to the cosine term.

Upon excitation of the room, some of the room modes will be set into oscillation thus sustaining the signal through the resonances. When the excitation is terminated, the resonances are left to decay due to the absorption within the room. This creates the reverberation field within the room.

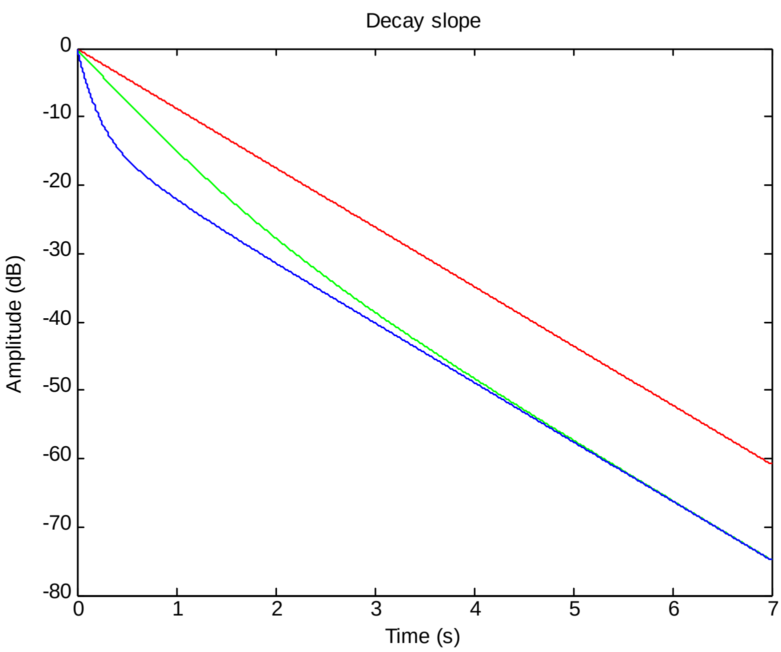
The reverberation time $T_{60}$ is defined as the time required for a decay of 60 dB, which corresponds to a reduction to 0.001 of the initial level, so the relation between one of the damping coefficients and the reverberation time for this particular oscillation becomes:

$$\exp(-\delta_n T_{60}) = 10^{-3} \quad \Rightarrow \quad T_{60} = \frac{-\ln(10^{-3})}{\delta_n} = \frac{6.91}{\delta_n}$$

The damping constant is directly related to wall impedance (see the end of section 2.3.5). Kuttruff states that the damping constant is within the range from 1 to 20 $s^{-1}$ for most rooms [K-83], which can be converted into a typical reverberation time within the range 0.35 and 7 s.

The result of plotting the expression for the sound pressure is shown below without the cosine in order to concentrate on the envelope. Three different combinations of the damping constants are in use and $N = 7$ generates 432 discrete frequencies from 34 to 476 Hz.

The top curve (red) uses constant damping constant of $\delta = 1\ s^{-1}$ for all frequencies and –60 dB is reached at the expected time mark. The middle curve (green) uses $\delta = 1\ s^{-1}$ for frequencies below 186 Hz and $\delta = 2\ s^{-1}$ for higher frequencies so the higher frequencies decay faster than the lower ones, which changes the slope. The bottom curve (blue) uses $\delta = 1\ s^{-1}$ below 185 Hz, the somewhat faster decay of $\delta = 3\ s^{-1}$ for up to 231 Hz, and $\delta = 9\ s^{-1}$ for higher frequencies. The high frequencies decay rapidly, which is shown by the rapid decay immediately after $t > 0$. The green and blue curves approach each other since the lowest part of the spectrum (below 186 Hz) is common to both curves when the high-frequency part has vanished.




**The decay slopes for different values of the damping constants. Red curve uses $\delta = 1\ s^{-1}$ for all frequencies. Green curve uses $\delta = 1\ s^{-1}$ below 186 Hz and $\delta = 2\ s^{-1}$ up to 486 Hz so the higher frequencies are decayed faster than the lower frequencies. Blue curve uses $\delta = 1\ s^{-1}$ below 186 Hz, just as the green curve, $\delta = 3\ s^{-1}$ up to 231 Hz and $\delta = 9\ s^{-1}$ up to 486 Hz.**

This ends the discussion of the solutions to the wave equation. The decay of the transfer function is exponential and the damping constant may vary among the resonance frequencies if the absorption material is not evenly distributed across the entire room surface, which it probably never is.

# 3. Reverberation

***Summary*** – The equation according to Sabine is derived from the assumption of a diffuse field, and used to calculate the distance where the direct sound from a source equates the sound within the diffuse field. The derivation is basically identical to the description by Kuttruff. The equation due to Eyring is derived using the mean free path length between reflections. An introduction is offered to the concept of mean free path length using a square room and the knowledge of mode distribution. The mathematics includes diffuse scattering and volume integrals, which was pretty hard to grasp, so material has been added, which was required for my understanding.

## 3.1. Sabine's formula

Wallace Clement Sabine initiated acoustical experiments at Harvard University and published 1890 the results of the empirical study of reverberation time, investigating the impact from absorption on the reverberation time. Sabine used a portable wind chest with organ pipes as the sound source and a stop watch determining the time from interruption of the source to inaudibility.

According to Sabine the reverberation time for a room is described through the volume $V$ in m$^3$ and an equivalent absorption area $A$ in m$^2$, which may consist of several areas $S_i$ each associated with an absorption coefficient $\alpha_i$. The reverberation time was found proportional to the ratio $V/A$.

$$T_{60} = 0.161\frac{V}{A}, \qquad A = \sum_i \alpha_i S_i$$

For a room with height 3 m, width 4 m and length 5 m, the volume is $V$ = 60 m$^3$. Assuming a floor carpet with $\alpha$ = 1 at the frequency of interest, the absorption area becomes $A$ = 12 m$^2$ and the reverberation time is estimated to $T_{60}$ = 0.8 s.

### 3.1.1. Derivation of Sabine's formula

An expression for the sound decay will be derived with onset in the assumption of a diffuse field within the room. A diffuse field can be regarded as consisting of countless contributions from plane waves arriving from all directions, so the power density of the diffuse field is found by integration through space, which is achieved by multiplication with 4π [K-129]. The resulting intensity of the diffuse field is reduced 4π times below that of a plane wave with the same energy density.

$$dw = \frac{I}{c}d\Omega \quad \Rightarrow \quad w = \frac{I}{c}\int_\Omega d\Omega = \frac{4\pi I}{c} \quad \Rightarrow \quad I = \frac{c}{4\pi}w$$

The energy incident on wall element $dS$ per second is estimated by integration of $I$ onto $dS$ through all directions within the room. The projection in the direction $\theta$ from the polar axis is $\cos(\theta)dS$, and if $I$ denotes intensity from that direction then $I\cos(\theta)dSd\Omega$ is the sound energy arriving per second on $dS$ from the solid angle $d\Omega$ [K-54]. The solid angle is $d\Omega = \sin(\theta)d\theta d\varphi$ using $\varphi$ for the azimuth angle [K-52]. The incident energy on $dS$ becomes [RW-168]:

$$E_i = \int_0^{2\pi}\int_0^{\pi/2} I\cos(\theta)\sin(\theta)dSd\theta d\varphi = IdS\int_0^{2\pi}d\varphi\int_0^{\pi/2}\cos(\theta)\sin(\theta)d\theta = IdS\,2\pi\left[-\frac{\cos^2(\theta)}{2}\right]_0^{\pi/2} = IdS\pi$$

The irradiation strength $B$ of a wall as the energy incident per unit time and unit area is [K-130]:

$$B = \pi I \quad \Rightarrow \quad B = \frac{c}{4}w$$

A sound source is supposed to feed the acoustical power $P$ into the room thus creating a balance with the rate of change of the energy density $Vw$ and the absorption, which has the absorption area $A$, and since the irradiation strength is $B$ the dissipated energy becomes $BA$.

$$P = V\frac{dw}{dt} + BA \quad \Rightarrow \quad P = V\frac{dw}{dt} + \frac{cA}{4}w$$

The sound source outputs constant power $P_0$ during the build-up of the sound pressure within the room, and it is assumed that the source has been applied for sufficient time to reach at a stable situation. The energy density is thus constant and the differential quotient is zero.

$$w_0 = \frac{4P_0}{cA}$$

After build-up, the sound source is interrupted, thus terminating power supplied to the system, and the decaying energy density is determined through solving the differential equation. This leads to an exponential decay with the time constant $4V/cA$ [RW-200]. The energy density must be continuous at $t = 0$, so the delay must start from $w_0$ immediately following the removal of the source.

$$0 = V\frac{dw}{dt} + \frac{cA}{4}w \quad \Rightarrow \quad w = w_0 \exp\left(-\frac{cA}{4V}t\right)$$

The reverberation time is specified as the time required for the energy density to reach one millionth of the initial level. Inserting the figure for the speed of sound we arrive at the reverberation formula according to Sabine with $V$ in $m^3$ and $A$ in $m^2$.

$$\exp\left(-\frac{cA}{4V}T_{60}\right) = 10^{-6} \quad \Rightarrow \quad \frac{cA}{4V}T_{60} = -\ln\left(10^{-6}\right) \quad \Rightarrow \quad T_{60} = \ln\left(10^{6}\right)\frac{4V}{cA} \quad \Rightarrow \quad \boxed{T_{60} = 0.161\frac{V}{A}}$$

***Measurement***

The energy density level was unavailable to Sabine, so he was forced to assume a fixed hearing threshold $w_{th}$ during testing and using at least two organ pipes to remove the energy density from the equation. One pipe generates the energy density $w_0$ and assuming uncorrelated sources, the level becomes $nw_0$ using $n$ pipes. Further assuming exponential decay, the slope constant becomes:

$$\begin{matrix} w_{th} = nw_0 \exp(-at_n) \\ w_{th} = w_0 \exp(-at_1) \end{matrix} \quad \Rightarrow \quad 1 = n\exp(-a(t_n - t_1)) \quad \Rightarrow \quad a = \frac{\ln(n)}{t_n - t_1}$$

$T_{60}$ is the time required for attenuation to one millionth of the initial energy density level.

$$\frac{w_{th}}{w_0} = \exp(-aT_{60}) = 10^{-6} \quad \Rightarrow \quad T_{60} = \frac{\ln\left(10^{-6}\right)}{-a} = \frac{\ln\left(10^{6}\right)}{\ln(n)}(t_n - t_1)$$

### 3.1.2. Diffuse field distance

The sound pressure from the loudspeaker will at some distance balance with the diffuse sound field within the room, and this distance is labelled the *diffuse field distance* or the *critical distance* $r_c$. The loudspeaker is assumed spherical, which limits the application to low frequencies[5].

[5] The loudspeaker becomes directive at $ka > 1$, with $a$ representing radius of the loudspeaker [B-103]. The assumption of spherical radiation requires $f < c/2\pi a$, so a typical loudspeaker with $a = 100$ mm is a point source below 550 Hz. The doubling of the sound pressure is due to diffraction and shall not be detailed here; see reference [S]. The increase in sound pressure is often counteracted by the manufacturer thus keeping the sound pressure level constant at the listening position thus reducing the power being output to the room.

The intensity of the sound $I_{LSP}$ at distance $r$ given by the acoustical output power $P_{LSP}$ distributed across the surface of a sphere with $r$ as radius and the corresponding energy density is calculated by division with the speed of sound [K-15].

$$I_{LSP} = \frac{P_{LSP}}{4\pi r^2} \quad \Rightarrow \quad w_{LSP} = \frac{I_{LSP}}{c} = \frac{P_{LSP}}{4\pi r^2 c}$$

The energy density within the room is $4P_0/cA$ as shown previously and with $P_{LSP}$ for the source, the critical distance follows immediately [K-147]. The critical distance is function of the equivalent absorption area since this is the only dissipative component, and does not depend upon the volume of the room. The equation can be changed to use the volume $V$ in m$^3$ and the reverberation time $T_{60}$ in seconds by using Sabine's formula for substitution with $A = 0.16\ V/T_{60}$.

$$w_0 = \frac{4P_0}{cA} \quad \wedge \quad P_0 = P_{LSP} \quad \Rightarrow \quad \frac{4P_{LSP}}{cA} = \frac{P_{LSP}}{4\pi r_C^2 c} \quad \Rightarrow \quad \boxed{r_C = \sqrt{\frac{A}{16\pi}} \approx 0.06\sqrt{\frac{V}{T_{60}}}}$$

For the previous room, with $V = 60$ m$^3$ the carpet was the only absorption area with $A = 12$ m$^2$ so the critical distance for the room becomes $r_C = 0.5$ m. This is a somewhat surprising result since the room corresponds roughly to a normal living room, where the direct sound does not appear dominated by the reverberation at significant distance from a loudspeaker. This impression must be due to the directivity of the loudspeaker, which directs the sound toward the listening place for middle to high frequencies while reducing the level within the diffuse field due to the reduced average power being output to the room. The reverberant field must anyway represent a significant level so the sound quality must be dependent upon the evenness of the reverberant field.

### 3.1.3. Sound pressure

The sound pressure within the diffuse field is identical to the sound pressure from the loudspeaker at the critical distance, thus enabling a determination of the overall sound pressure within the room. The sound intensity from the loudspeaker is defined from the root mean sound pressure $p_{rms}$ [K-15], so the sound pressure within the diffuse field becomes:

$$I_{LSP} = \frac{p_{rms}^2}{\rho_0 c} = \frac{P_{LSP}}{4\pi r^2} \quad \Rightarrow \quad p_{rms} = \frac{1}{r_C}\sqrt{\frac{\rho_0 c P_{LSP}}{4\pi}} = \sqrt{\frac{16\pi}{A}}\sqrt{\frac{\rho_0 c P_{LSP}}{4\pi}} \quad \Rightarrow \quad \boxed{p_{rms} = \sqrt{\frac{4\rho_0 c P_{LSP}}{A}}}$$

For an acoustical output of $P_{LSP} = 1$ W from the loudspeaker within the above room with $A = 12$ m$^2$, the sound pressure within the diffuse field becomes $p_{rms} = 3.7$ Pa, corresponding to 115 dB above 20 µPa. It should be noted that the efficiency of typical loudspeakers is 1 %, defined as the amount of electrical power which is being output as acoustical power, so the loudspeaker needs a power amplifier capable of driving 100 W into the loudspeaker without distortion of the signal.

### 3.1.4. Limitations

The assumption of a perfect diffuse field is not met in real life; the sound field must include a source of energy since the energy is dissipated at the surface. Typically the source is a loudspeaker located close to a corner thus sourcing energy at one point of the room while the dissipation takes place at the surface area. There is thus a net flow of energy through the room so the field cannot be perfectly diffuse.

Another problem with Sabine's equation is related to rooms with high absorption. Consider for instance a room with total absorption for the entire surface ($\alpha = 1$ and hence $A = S$). This room is free of reverberation since there is no reflection from any part of the surface and the reverberation time is zero. However, according to Sabine's formula the estimated reverberation time is non-zero, so a correction of the formula is required for rooms with significant absorption.

## 3.2. Eyring's formula

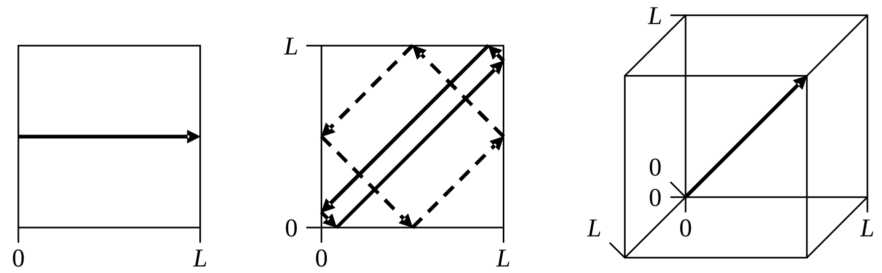
An improved equation for calculation of the reverberation time is Eyring's formula [K-140], which accepts large absorption coefficients. The volume of the room is $V$ in m$^3$ and the total surface area is $S$ in m$^2$. The surface may consist of different parts with area $S_i$ for the section with the absorption coefficient $\alpha_i$. The formula includes the absorption due to air through the parameter $m$, with an approximate value of $1.1\cdot10^{-3}$ m$^{-1}$ at 1 kHz, 20°C and 60 % of relative humidity [K-163].

$$T_{60} = 0.161 \frac{V}{4mV - S\ln(1-\bar{\alpha})} \qquad \bar{\alpha} = \frac{1}{S}\sum_i \alpha_i S_i$$

The denominator approaches infinity for large value of the absorption coefficient thus reducing the reverberation time toward zero. For low absorption the logarithm becomes $\ln(1-\alpha) \approx -\alpha$ [RW-197], so the formula approaches Sabine's formula for rooms with moderate absorption.

### 3.2.1. Introduction to the derivation

The derivation uses the *mean free path length*, which represents the average distance a wave travels between reflections, and an approximate derivation illustrates the method using a cube with side $L$.




**The mean free path length is approximated using a cube as the room model.**

A plane wave travelling along one of the axes moves length $L$ before being reflected, so the mean free path length becomes $l_1 = L$. However, the wave may bounce through four surfaces with a total path length of $2\cdot(L^2 + L^2)^{0.5} = 2.82\, L$, and it hits the boundary four times so $l_2 = 0.707\, L$. The wave may also move between all sides, with a length from corner to corner of $2\cdot(L^2 + L^2 + L^2)^{0.5} = 3.46\, L$, and with six sides we get $l_3 = 0.577\, L$. Using the distribution of modes known from section 2.3.2, an average value using weights of 1:6:12 produces: $l = (1 + 6\cdot0.707 + 12\cdot0.577)/19 = 0.64\, L$. The mean number of reflections per second becomes $c/l$, and by multiplication with $L^2/L^2$ the volume becomes $V = L^3$ and the surface area is $S = 6L^2$. The total number of reflections up to time $t$ is estimated.

$$\bar{n} = \frac{c}{0.64L} = \frac{cL^2}{0.64L^3} = \frac{1}{6}\frac{cS}{0.64V} = \frac{cS}{3.84V} \quad \text{and} \quad N = \bar{n}t = \frac{cS}{3.84V}t$$

The energy decays due to the absorption coefficient $\alpha$ with a reduction of $1-\alpha$ for each of the $N$ reflections and this may be written as an exponential decay function by introducing the logarithm.

$$E_N(t) = E_0(1-\alpha)^N = E_0 \exp(N\ln(1-\alpha)) = E_0 \exp\left(\frac{cS}{3.84V}\ln(1-\alpha)t\right)$$

At the reverberation time $t = T_{60}$ the energy has decayed to $10^{-6}$, and we arrive at an equation, which is fairly close to Eyring's formula (4 % on the low side and without air absorption).

$$\exp\left(\frac{cS\ln(1-\alpha)}{3.84V}T_{60}\right) = 10^{-6} \quad \Rightarrow \quad T_{60} = \frac{\ln(10^{-6})\cdot 3.84V}{cS\ln(1-\alpha)} = 0.155\frac{V}{S\ln(1-\alpha)}$$

### 3.2.2. Derivation of Eyring's formula

The speed of sound is *c* so the wave travels the distance *ct* within the time span *t*. The number of reflections during this time interval is *N* and this leads to the average reflection frequency.

$$\bar{l} = \frac{ct}{N} = \frac{c}{\bar{n}} \quad \text{where} \quad \bar{n} = \frac{N}{t}$$

All paths are not traversed with equal probability; rather the chance is greater for the cords close to the normal according to Lambert's law, which states that the perfectly diffuse reflection is weighted by $\cos(\upsilon)/\pi$, where $\upsilon$ is the angle of the beam from the direction of the normal and $R_{ds}$ is the length of the cord from the infinitesimal surface area. The law does not care for the angle of the irradiating source, so there is no information of the past [K-122].

$$l(\upsilon, \Omega) = \frac{\cos(\upsilon)}{\pi} R_{dS} d\Omega$$

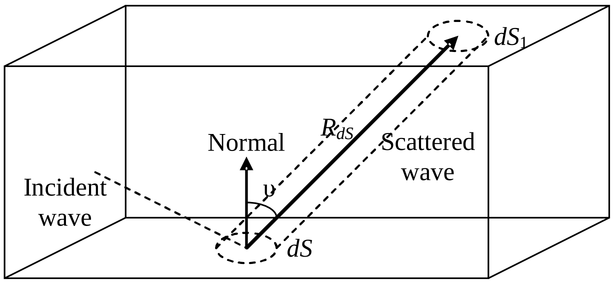
The probability of a direction is the largest for $\upsilon$ close to zero, where $\cos(\upsilon)$ approaches one, and is reduced toward zero for $\upsilon$ close to $\pi/2$. The length of the cord is averaged by integration over all possible cords within the room emerging from the surface element *dS*, which is integration through the internal space *Ω* of the room, and over all of the surface area, which is integration through *S*.

$$\bar{l} = \frac{1}{S}\int_S\int_\Omega l(\upsilon, \Omega) d\Omega dS = \frac{1}{S}\int_S\int_\Omega \frac{\cos(\upsilon)}{\pi} R_{dS} d\Omega dS$$

The volume integral will be considered first. The integral of $\cos(\upsilon)$ through all directions within the room uses $\upsilon$ from 0 to $\pi/2$ and anticlockwise around the normal from 0 to $2\pi$. Integration of the cosine from 0 to $\pi/2$ produces the value one, and integration through the circle corresponds to $2\pi$. After division with $\pi$ from the denominator the figure of two is returned from the integration. This leaves the problem of the integration of $R_{dS}$ through the entire surface, which will be explained with reference to the illustration below.

$$\bar{l} = \frac{2}{S}\int_S R_{dS} dS = \frac{4V}{S}$$

The product $R_{ds}dS$ is the volume of an infinitesimal tube with length $R_{ds}$ extending from surface to surface, and with the cross sectional area *dS*. The integration results in two times the volume *V* of the room, since the cord from *dS* to $dS_1$ represents the same volume when the points are being interchanged during integration, so all volume is included two times. Including the factor of two from the previous integration the result is 4*V* divided by the total surface *S* of the room [K-135].




**Perfectly diffuse scattering is not related to the angle of the incoming sound wave. The reflection may have any direction within the room with a weighting function according to Lambert stating the**

**probability of a given direction with the normal being the most probable (υ = 0) and the direction along the surface being the least probable.**

Using the initial definition of the average reflection frequency and the mean free path length gives an expression for determining the value of the average reflection frequency [K-135].

$$\bar{l} = \frac{c}{\bar{n}} = \frac{4V}{S} \quad \Rightarrow \quad \bar{n} = \frac{cS}{4V}$$

The surface is assumed build from two parts: $S_1$ with the absorption coefficient $\alpha_1$ and $S_2$ with the absorption coefficient $\alpha_2$, and where $S = S_1 + S_2$. For a sound ray hitting the surface a total of $N$ times, there will be $N_1$ hits on surface $S_1$ and $N - N_1$ hits on surface $S_2$ and the numbers will be distributed in some way about their mean values, shown below for $N_1$ and similarly for $N_2$. After $N_1$ reflections from surface $S_1$ and $N - N_1$ reflections from surface $S_2$, the energy has been reduced from an initial level $E_0$ to the energy level $E_N(N_1)$ [K-138].

$$\bar{N}_1 = N\frac{S_1}{S} \quad \text{and} \quad E_N(N_1) = (1-\alpha_1)^{N_1}(1-\alpha_2)^{N-N_1}E_0$$

Since there is no record of the previous sound ray, the reflections are stochastically independent, and the probability of $N_1$ reflections with surface $S_1$ is given by the binominal distribution[6].

$$P_N(N_1) = \binom{N}{N_1}\left(\frac{S_1}{S}\right)^{N_1}\left(\frac{S_2}{S}\right)^{N-N_1} \quad \text{where} \quad \binom{N}{N_1} = \frac{N!}{N_1!(N-N_1)!}$$

The expected value for a discrete random variable is the sum of products of energies $E_N(N_1)$ and their probability $P_N(N_1)$ through for all possible values of $N_1$ [RW-428]. The expressions for energy and probability are inserted into the sum and the terms are rearranged [K-139].

$$\langle E_N \rangle = \sum_{N_1=0}^{N} E_N(N_1)P_N(N_1) = \sum_{N_1=0}^{N}\binom{N}{N_1}\left(\frac{S_1}{S}(1-\alpha_1)\right)^{N_1}\left(\frac{S_2}{S}(1-\alpha_2)\right)^{N-N_1}E_0$$

Using the binomial theorem the sum with $N_1$ from zero to $N$ can be substituted with a much simpler expression, which leads to a fairly useful expression of the energy level after $N$ hits [RW-44].

$$\sum_{k=0}^{N}\binom{N}{k}a^k b^{N-k} = (a+b)^N \quad \Rightarrow \quad \langle E_N \rangle = \left(\frac{S_1}{S}(1-\alpha_1) + \frac{S_2}{S}(1-\alpha_2)\right)^N E_0 = \left(1 - \frac{\alpha_1 S_1}{S} - \frac{\alpha_2 S_2}{S}\right)^N E_0$$

The two last terms of the parenthesis can be combined into the arithmetic mean of the absorption coefficients. The power expression is rewritten using the logarithm. It is obvious that the average absorption coefficient can be expanded to the mean of any large number of surfaces.

$$\langle E_N \rangle = (1-\bar{\alpha})^N E_0 = \exp(N\ln(1-\bar{\alpha}))E_0 \quad \text{where} \quad \bar{\alpha} = \frac{\alpha_1 S_1 + \alpha_2 S_2}{S}$$

Finally the total number of reflections $N$ is substituted by the average reflection frequency $cS/4V$ multiplied by the time duration $t$ for the total number of reflections.

$$N = \bar{n}t = \frac{cS}{4V}t \quad \Rightarrow \quad E(t) = E_0 \exp\left(\ln(1-\bar{\alpha})\frac{cS}{4V}t\right)$$

[6] Wikipedia: The binomial distribution is the discrete distribution of the number of successes in a sequence of yes/no experiments, each of which yields probability of $p$. The binomial coefficient, which may also be written $C(n,k)$, is the number of ways we can get $k$ successes with the probability $p^n$ and $n - k$ failures with the probability $(1 - p)^{n-k}$.

The wave front undergoes attenuation during propagation through air, and this can be represented by an exponential decay related to the distance *ct* being travelled between the reflections according to section 2.1.3.

$$E(t) = E_0 \exp\left( \ln(1-\overline{\alpha}) \frac{cS}{4V} t \right) \exp(-mct) = E_0 \exp\left( \left( \ln(1-\overline{\alpha}) \frac{cS}{4V} - mc \right) t \right)$$

The reverberation time $T_{60}$ can now be determined as the time for reduction of the energy by 60 dB to one millionth of the initial energy level.

$$\exp\left( \left( \ln(1-\overline{\alpha}) \frac{cS}{4V} - mc \right) T_{60} \right) = 10^{-6} \quad \Rightarrow \quad \left( \ln(1-\overline{\alpha}) \frac{cS}{4V} - mc \right) T_{60} = \ln\left(10^{-6}\right)$$

Solving for the reverberation time the result is a constant times the ratio of the volume to an expression with the surface area, the average absorption coefficient and the air absorption.

$$T_{60} = \frac{\ln\left(10^{-6}\right)}{\ln(1-\overline{\alpha}) \frac{cS}{4V} - mc} = \frac{4\ln\left(10^{-6}\right)}{c} \frac{V}{S\ln(1-\overline{\alpha}) - 4mV}$$

By insertion of the speed of sound we arrive at Eyring's formula, with the volume *V* in m$^3$ and the surface area *S* in m$^2$. This formula has been developed independently by Norris as well as Schuster and Waetzmann.

$$T_{60} = 0.161 \frac{V}{4mV - S\ln(1-\overline{\alpha})} \quad \text{where} \quad \overline{\alpha} = \frac{1}{S}\sum_i \alpha_i S_i$$

### 3.2.3. Large room

The additional term represents air absorption and is significant only when $4mV$ is large compared to the equivalent absorption area $\alpha S$ (it has been assumed that $\alpha$ is so small that the logarithm can be approximated by the equivalent absorption coefficient).

$$V > \frac{\overline{\alpha} S}{4m}$$

As shown by the examples, the volume should be relatively large before air absorption becomes crucial. A typical lecture room is so small that air absorption can be ignored, but a concert hall may well exceed the limit thus requiring the correction.

For $S = 100$ m$^2$, $m = 20 \cdot 10^{-3}$ m$^{-1}$ (8 kHz) and very modest absorption of $\alpha = 0.1$ the air absorption should be considered at high frequencies for $V > 500$ m$^3$.

For a cube with side length *L* we have $V = L^3$ and $S = 6L^2$ so the requirement can be related to the required side length of the cube. Using the highest value of m = $20.5 \cdot 10^{-3}$ m$^{-1}$ (8 kHz) and very modest absorption of $\alpha = 0.1$ we get a requirement of $L > 7.3$ m corresponding to at least $V = 390$ m$^3$ and $S = 54$ m$^2$.

$$L^3 > \frac{6L^2\overline{\alpha}}{4m} \quad \Rightarrow \quad L > \frac{1.5\overline{\alpha}}{m}$$

# 4. MATLAB code

The pictures in section 2.3 and the conclusion to the distribution of modes within section 2.3.2 were generated using the following code. Input variables are the room measures (height, width and length) and the maximum number for the modes. The vector f carries the resonance frequencies of the rectangular room and is reordered after calculation with f(1) for the lowest frequency. The vector g carries the frequency difference.

```
% RoomResonances.m
clear all
Lx= 3;  % Height (m).
Ly= 4;  % Width  (m).
Lz= 5;  % Length (m).
c=343;  % Speed of sound (m/s).
N=  7;  % Number of eigenvalues for each dimension.
f=zeros(1,N^3-1);
g=zeros(1,N^3-1);
z0=0; z1=0; z2=0; z3=0;
% Plot the resonance frequencies.
for nx=0:N-1
    for ny=0:N-1
        for nz=0:N-1
            if (nx+ny+nz==0)
                z3=z3+1;
            else
                if (nx+ny==0) || (nx+nz==0) || (ny+nz==0)
                    z2=z2+1;
                else
                    if (nx==0) || (ny==0) || (nz==0)
                        z1=z1+1;
                    else
                        z0=z0+1;
                    end
                end
            end
            n=nx+N*ny+N^2*nz;
            if n>0
                f(n)=(c/2)*sqrt((nx/Lx)^2+(ny/Ly)^2+(nz/Lz)^2);
            end
        end
    end
end
figure(1)
f=sort(f);
plot(f,'-b')
title('Resonance frequencies for room with H=3m W=4m L=5m')
xlabel('Mode n')
ylabel('Frequency (Hz)')
% Plot the difference frequencies.
for n=1:length(f)-1
    g(n)=(f(n+1)-f(n))/2;
end
figure(2)
plot(g,'-r')
title('Frequency difference for room with H=3m W=4m L=5m')
xlabel('Mode n')
ylabel('Difference frequency (Hz)')
disp(['Mean difference = ' num2str(mean(g))])
disp(['N=' num2str(N) ', z3=' num2str(z3) ', z2=' num2str(z2) ...
      ', z1=' num2str(z1) ', z0=' num2str(z0) ...
      ', sum=' num2str(z0+z1+z2+z3)])
```

The picture in section 2.3.6 was generated from the following code, which uses the frequency vector f being output from the previous code. The time scale t ranges from 0 to 7 s with 1 ms of time increment. Two constants are defined with n1 set to one-fifth of the total number of modes (the lowest 20 % of the resonance frequencies) and n2 set to one-third of the total number of modes (the lowest 33 % of the resonance frequencies). The constants are used to set the damping constant d during execution and variable m is used to generate three curves with individual .

```
% ReverberationDecay.m
% Uses the frequency vector "f" from RoomResonances.m
t=0:0.001:7;            % Time axis (s).
figure(3)
n1=length(f)/5;
n2=length(f)/3;
for m=1:3
    p=zeros(1,length(t));
    for n=1:length(f)
        d=(m==1)+ ...
          (m==2)*((n<n1)+2*(n>=n1))+ ...
          (m==3)*((n<n1)+3*(n>=n1)+6*(n>n2));
        p=p+exp(-d*t)/length(f);
    end
    if (m==1) plot(t, 20*log10(abs(p)), '-r'); end
    if (m==2) plot(t, 20*log10(abs(p)), '-g'); end
    if (m==3) plot(t, 20*log10(abs(p)), '-b'); end
    hold on
end
hold off
title('Decay slope')
xlabel('Time (s)')
ylabel('Amplitude (dB)')
```

## 5. References

The following books are referenced using either K or CA followed by the page number. Although the Internet cannot be used as a trusted scientific reference, the *Uniform Resource Locator* (URL), i.e. the Internet “address” will be used for lightweight references.

B: Leo Beranek *Acoustics*, Reprinted 1996, Acoustical Society of America.

K: Heinrich Kuttruff *Room Acoustics*, 5th Ed. 2009, Spon Press.

RW: Lennart Råde and Bertil Westergreen *Mathematics Handbook*, 5th Ed. 2004, Studentlitteratur.

S: Tore Skogberg *Loudspeaker Cabinet Diffraction*, Master project, 2006, DTU. Available for download: http://www.torean.dk/artikel/Diffraction.pdf.