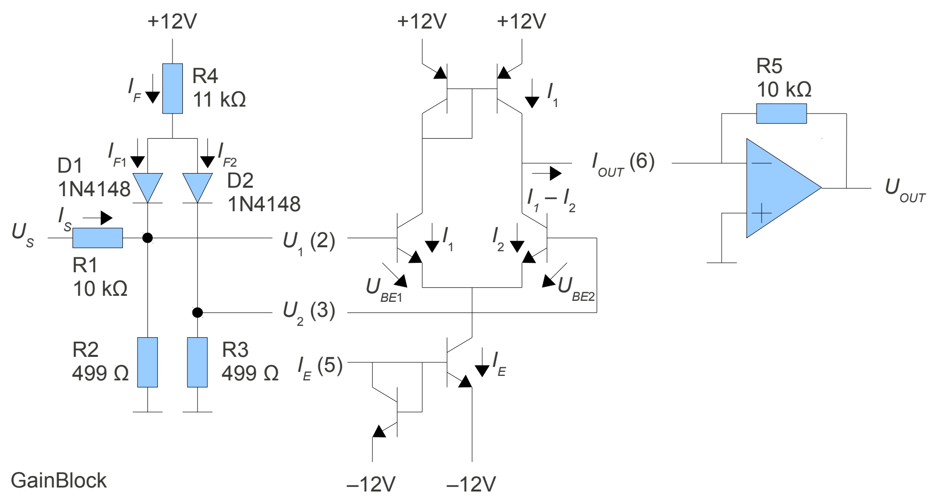
Gain control

An electronically adjustable gain stage using the LM3080 device[1] is analysed for the mechanism behind the gain control deriving the design equations for signal gain, distortion and pre-distortion circuitry. The circuit was used for volume control within an electronic organ and the following analysis was required to define the allowable signal levels for inaudible distortion.

Transconductance amplifier

The LM3080 device is a *transconductance* amplifier, which means that the input signal is *voltage* while the output signal is *current* through the relation $I_{OUT} = gU_{IN}$ where g is the proportionality parameter between input and output with the unit of ampere/volt (A/V), which is called siemens (S) in Europe and “mho” in America. A simplified schematic of the LM3080 device is shown below (middle part) without details referring to the level-shifting circuitry (current mirrors); however the schematic contains the important part of the circuitry. Also shown is the conventional output buffer using an operational amplifier with R_5 for conversion of I_{OUT} into voltage (*transresistance* amplifier). To the left is shown the pre-distortion circuit to be analysed later in this document.



The gain block uses the current-dependency of the transconductance and identical transistors are used to remove the temperature dependency and the control current. An operational amplifier buffers the current back into voltage. The diode network shown left increases the linear range.

The heart of the gain block is the differential stage (common emitter) with input at pin 2 and 3. The stage is controlled by a current mirror directing the current I_E at pin 5 through the transistors, which divides the current into I_1 and I_2 in relation to the input differential voltage. The current I_1 is mirrored and routed to the output where the current I_2 is subtracted. The following is obvious:

$$I_E = I_1 + I_2$$
$$I_{OUT} = I_1 - I_2$$

The input voltage difference ΔU is the voltage potential between the input voltages U_1 and U_2 with zero difference at equilibrium. The base-emitter diodes of the transistors are connected in series (or anti series due to the sign convention) from U_1 to U_2 so the voltage across the series connected diodes is defined as follows.

$$\Delta U = U_1 - U_2 = U_{BE1} - U_{BE2}$$

1 Unfortunately the device has been discontinued by National but the LM13600 (dual LM3080) is still available.

The collector current of a transistor is defined by the voltage across the base-emitter diode through an exponential relation[2].

$$I_1 = I_0 \exp(\frac{U_{BE1}}{U_T}) \qquad I_2 = I_0 \exp(\frac{U_{BE2}}{U_T}) \quad \text{where} \quad U_T = \frac{kT}{q} \quad \text{and} \quad \begin{aligned} k &= 1.38 \cdot 10^{-23}\,\text{J/K} \\ T &= \theta + 273.15\,\text{K} \\ q &= 1.602 \cdot 10^{-19}\,\text{C} \end{aligned}$$

The current I_0 is defined by the manufacturing process and is around 10 fA for small-signal devices. The temperature coefficient is 10 % per degree centigrade so the current doubles at 7°C rise. This dependency is compensated using two transistors with equal I_0, which is a realistic assumption for an integrated circuit but probably not using discrete transistors. The *temperature voltage* U_T is given by Boltzmanns constant *k,* the unit charge *q* and the absolute temperature *T*. The value is found to U_T = 26 mV at *T* = 300 K (27°C) where the temperature coefficient is 0.33 %/K.

The ratio of the output current to the emitter current removes the troublesome parameter I_0 when the above exponential relation is introduced.

$$\frac{I_{OUT}}{I_E} = \frac{I_1 - I_2}{I_1 + I_2} = \frac{\exp(\frac{U_{BE1}}{U_T}) - \exp(\frac{U_{BE2}}{U_T})}{\exp(\frac{U_{BE1}}{U_T}) + \exp(\frac{U_{BE2}}{U_T})}$$

At equilibrium the differential input voltage is $\Delta U = 0$ so $U_{BE1} = U_{BE2}$ and this value will be called U_{BE}. Assuming that the differential voltage is divided equally between the transistors, we have:

$$U_{BE1} = U_{BE} + \frac{\Delta U}{2} \quad \text{and} \quad U_{BE2} = U_{BE} - \frac{\Delta U}{2}$$

This is inserted into the above ratio and the common factor is removed by division.

$$\frac{I_{OUT}}{I_E} = \frac{\exp(\frac{\Delta U}{2 U_T}) - \exp(\frac{\Delta U}{2 U_T})}{\exp(\frac{\Delta U}{2 U_T}) + \exp(\frac{\Delta U}{2 U_T})}$$

This can be expressed using the hyperbolic tangent [1-122]; a smooth function, which moves from the lower asymptotic value –1 at large negative argument to the upper asymptotic value +1 at large positive argument and crosses through zero at zero input. Hence, the output current is the emitter current multiplied by the hyperbolic function of the differential input voltage normalised by $2U_T$.

$$I_{OUT} = I_E \tanh(\frac{\Delta U}{2 U_T}) \quad \text{where} \quad 2 U_T \approx 52\,\text{mV}$$

Around zero the hyperbolic tangent function approximates the argument: tanh(*x*) ≈ *x* [1-197], which leads to the following relation. The correctness of the assumption will be analysed later but for now the amplitude of the input signal must be limited to few millivolt so that *x* approaches zero.

$$I_{OUT} = g \Delta U \quad \text{where} \quad g = \frac{I_E}{2 U_T} \quad \text{assuming} \quad \Delta U \ll 2 U_T$$

At I_E = 500 µA the transconductance becomes *g* = 9.6 mS, which is right at the published value for LM3080 and LM13600 according to the respective data sheets. Hence, for ΔU = 1 mV input offset between the differential inputs the output current becomes I_{OUT} = 9.6 µA.

2 See for instance the Ebers-Moll transistor model.

The output current is routed to an inverting amplifier with feedback resistor R_5 so the output voltage becomes proportional to the product of the emitter current I_E and the input voltage ΔU, which is the core of the gain control; the voltage gain is gR_5 where g is current controlled.

$$U_{OUT} = -R_5 I_{OUT} = -gR_5 \Delta U$$

For R_5 = 10 kΩ and I_E = 1 mA we have g = 19.3 mS so the gain is 193 times and unity-gain needs a current of 5.2 µA. The relation is valid for at least three decades for LM3080 and is specified to six decades for LM13600; typically from 1 nA up to 1 mA so the gain regulation range is 120 dB. It is downward limited by thermal noise and leakage current; and upward limited by the resistive part of the base junction and by thermal effects related to the power being dissipated.

The input signal must typically be reduced to the millivolt level to ensure linearity so resistor R_1 is added to reduce the signal by voltage division with R_2. Assuming $R_1 >> R_2$ the reduction is R_2/R_1 so the relation between input voltage U_S and U_{OUT} becomes:

$$\boxed{U_{OUT} \approx -\frac{R_2}{R_1}\frac{I_E R_5}{2U_T}U_S}$$

For unity gain at I_E = 1 mA the requirement is $U_S/\Delta U$ = 193 so the attenuator must be designed with $R_1 \approx 193\ R_2$ (or $R_1 \approx 192\ R_2$ to be exact) and using R_2 = 499 Ω the value becomes R_1 = 96 kΩ.

Non-linearity

The differential pair is assumed linear by requiring ΔU being “small” but this criteria is rather vague. A preferred method is to define an acceptable level of non-linearity, i.e. a measure of the deviation from a straight line and one approach is to represent the hyperbolic tangent by series-expansion according to Taylor [1-197]. The first-order term represents the proportionality factor corresponding to distortion-free operation and terms of higher order represents the distortion. An acceptable level of distortion of 1 % will be assumed[3] and using only the two first terms (x and $x^3/3$) the requirement is that the latter term is no more than 1 % of the former. This truncation of the series is acceptable since the high-order terms decay fast for $x < 1$.

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots \quad \Rightarrow \quad \frac{x^3}{3} < 0.01\,x \quad \Rightarrow \quad x < \sqrt{0.03} = 0.173$$

The requirement for less than 1 % non-linearity is an input signal below 9 mV.

$$x = \frac{|\Delta U|}{2U_T} < 0.173 \quad \Rightarrow \quad |\Delta U| < 0.173 \cdot 2U_T \approx 9\,\mathrm{mV}$$

With a sine wave as input, the acceptable level is approximately two times higher as will be shown using m for the amplitude of a sine waveform and with the hyperbolic tangent again approximated by the first two terms.

$$x = \frac{\Delta U}{2U_T} = m\sin(\omega t) \quad \Rightarrow \quad \tanh(x) \approx x - \frac{x^3}{3} = m\sin(\omega t) - \frac{m^3}{3}\sin^3(\omega t)$$

The sine cube is expanded using a table of trigonometric formulas [1-128]:

$$\tanh(x) \approx m\sin(\omega t) - \frac{m^3}{3}\left[\frac{3\sin(\omega t) - \sin(3\omega t)}{4}\right] = \left(m - \frac{m^3}{4}\right)\sin(\omega t) + \frac{m^3}{12}\sin(3\omega t)$$

The first term shows that the fundamental component is compressed at high level since m is reduced by the term $m^3/4$. The harmonic distortion is the fraction between the amplitude of the term

3 This is the traditional engineering compromise.

oscillating with $3\omega t$ and the amplitude of the term oscillating with ωt. Assuming that m is less than unity the $m^3/4$ term can be ignored and the distortion relation becomes quadratic as shown below. For 1 % of third-harmonic distortion ($D_3 = 0.01$) the input level should be limited to $m = 0.35$.

$$D_3 = \frac{m^3}{12(m - m^3/4)} \approx \frac{m^2}{12} \quad \Rightarrow \quad m = \sqrt{12\,D_3} = \sqrt{12 \cdot 0.01} = 0.35$$

For a sine wave input the signal amplitude becomes limited to 18 mV (or 13 mV RMS).

$$x = \frac{\Delta U}{2\,U_T} = 0.35 \sin(\omega t) \quad \Rightarrow \quad \Delta U = U_0 \sin(\omega t) \quad \text{so} \quad U_0 = 0.35 \cdot 2\,U_T \approx 18\,\text{mV}$$

Using input attenuation giving unity gain at I_E = 1 mA an attenuation of 193 times is required and the allowable input amplitude becomes ±3.5 V corresponding to 2.6 V RMS. With this input level the output signal ±3.5 V with R_5 = 10 kΩ, I_E = 1 mA, R_1 = 96 kΩ and $R_2 = R_3$ = 499 Ω.

Pre-distortion

Diodes D_1 and D_2 are used to extend the useful input range by introducing a controllable amount of distortion of the signal, which counteracts the compression from the hyperbolic tangent. The diodes are biased at 1 mA through R_4 and the operation will be detailed. The forward current I_F is divided into the diode currents I_{F1} and I_{F2} respectively.

$$I_F = I_{F1} + I_{F2}$$

The current flows through resistors R_2 and R_3 thus giving an input common-mode of 250 mV with the proposed values. An input offset voltage (differential input) results if the resistors are not equal so precision resistors of ±0.1 % are recommended (giving ±0.25 mV worst-case offset) and there should not be applied a DC voltage so an input capacitor is recommended. With standard resistors (±1 %) the input offset is ±2.5 mV so the output offset becomes ±48 µA at I_E = 1 mA and the output DC reaches ±0.48 V at R_5 = 10 kΩ (proportional to the gain setting).

The node at U_1 receives an input current I_S from the input signal, which is given by the following approximate relation assuming $R_1 \gg R_2$.

$$I_S = \frac{U_S}{R_1} \quad \text{where} \quad R_1 \gg R_2$$

The input voltages U_1 and U_2 are generated by the currents flowing through the resistors R_2 and R_3 to be represented by R_2 = 499 Ω at both nodes since $R_2 = R_3$. The currents into node U_1 are I_S and I_{F1} and flows through R_2 and the current into node U_2 is I_{F2} flowing through $R_3 = R_2$. The differential input voltage to the differential pair becomes as follows.

$$\Delta U = U_1 - U_2 = (R_2 I_S + R_2 I_{F1}) - (R_2 I_{F2})$$

The difference between the diode currents will be normalised by the sum of the currents and this is allowable if at the same time multiplying with the sum (equal to I_F) thus not changing anything.

$$\Delta U = R_2 I_S + \frac{I_{F1} - I_{F2}}{I_{F1} + I_{F2}} R_2 I_F$$

The exponential relation for the transistor applies equally well to diodes (the sole difference is the base current, which is around 1 % of the collector current) so the diode forward current I_{F1} and I_{F2} is function of the forward voltage drop U_{F1} and U_{F2} using the same equation.

$$I_{F1} = I_0 \exp\left(\frac{U_{F1}}{U_T}\right) \quad \text{and} \quad I_{F2} = I_0 \exp\left(\frac{U_{F2}}{U_T}\right)$$

It will be assumed that the diodes are sufficient equal so I_0 can be used for both diodes; this may have some impact using discrete diodes (1N4148) with their unmatched parameters but this issue will not be the subject of further study. These expressions are substituted into the equation for *ΔU*.

$$\Delta U = R_2 I_S + \frac{\exp(\frac{U_{F1}}{U_T}) - \exp(\frac{U_{F2}}{U_T})}{\exp(\frac{U_{F1}}{U_T}) + \exp(\frac{U_{F2}}{U_T})} R_2 I_F$$

At equilibrium we have $U_{F1} = U_{F2}$, which will be called U_F and assuming that the differential voltage is divided equally between the diodes we have:

$$U_{F1} = U_F - \frac{\Delta U}{2} \quad \text{and} \quad U_{F2} = U_F + \frac{\Delta U}{2}$$

Insertion and removal of the common term due to U_F, and identifying the ratio of the exponentials as *minus* the hyperbolic tangent, we get:

$$\Delta U = R_2 I_S + \frac{\exp(\frac{-\Delta U}{2U_T}) - \exp(\frac{\Delta U}{2U_T})}{\exp(\frac{-\Delta U}{2U_T}) + \exp(\frac{\Delta U}{2U_T})} R_2 I_F \quad \Rightarrow \quad \Delta U = R_2 I_S - \tanh(\frac{\Delta U}{2U_T})$$

The negative sign is the important feature of the design; it means that the differential voltage *ΔU* will be enlarged at increasing input since the hyperbolic tangent adds the most around zero while less is added at increased level thus compensating for the signal compression.

$$\Delta U + R_2 I_F \tanh(\frac{\Delta U}{2U_T}) = R_2 I_S$$

However, the differential input signal is defined by a transcendent equation so it is not possible to present an exact analytical solution and another approach must be used. The derivation to follow is somewhat lengthy so the equation will be simplified to use single variables through normalisation with $2U_T$ for the differential voltage and through the introduction of I_F into the R_2I_S term.

$$\frac{\Delta U}{2U_T} + \frac{R_2 I_F}{2U_T} \tanh(\frac{\Delta U}{2U_T}) = \frac{R_2 I_F}{2U_T} \frac{I_S}{I_F}$$

Here *v* is the normalised differential input voltage, *s* is the normalised input current and constant *A* represents a reduction of the input signal caused by the resistance R_2 (and R_3) and the incremental resistance of the pre-distortion diodes, which can be calculated according to $r_D = U_T/I_{F1} \approx 50\ \Omega$ for each diode so there is in effect an incremental (non-linear) resistance of 100 Ω shunted across the input. The constant is $A = 9.6$ with the nominal values $U_T = 26$ mV, $R_2 = 499\ \Omega$ and $I_F = 1$ mA.

$$v = \frac{\Delta U}{2U_T} \quad \text{and} \quad A = \frac{R_2 I_F}{2U_T} \approx 9.6 \quad \text{and} \quad s = \frac{I_S}{I_F}$$

The substitutions simplifies the equation into:

$$v + A\tanh(v) = As \quad \text{where} \quad |v| < 1, \quad |s| < 1, \quad A > 1$$

For *v* sufficiently small the hyperbolic tangent is approximated by the argument through $\tanh(v) \approx v$ so the relation becomes $v \approx as$ where *a* is introduced to reduce the writing for the following steps.

$$v \to 0 \quad \Rightarrow \quad v + Av = As \quad \Rightarrow \quad v = as \quad \text{where} \quad a = \frac{A}{A+1}$$

For large values of *v* the hyperbolic tangent is approximated through the Taylor series-expansion. The are terms expressed in *v* but the high-order terms can safely be substituted by $v \approx as$ without introducing too much error since the higher-order terms are merely corrections to an approximate relation with $\tanh(v) \approx v$ where *v* is the important factor since distortion is assumed low; however, it is not approaching zero. The transcendent equation becomes:

$$v + A\left[v - \frac{v^3}{3} + \frac{2v^5}{15} - \ldots\right] = As \underset{v \approx as}{\Rightarrow} v + A\left[v - \frac{(as)^3}{3} + \frac{2(as)^5}{15} - \ldots\right] \approx As$$

Assembling *v* on the left side and *s* on the right side, dividing by *A* + 1 and introducing parameter *a* for the new term with *A* divided by *A* + 1 produces the following relation between *v* and *s*.

$$(A+1)v = A\left[s + \frac{(as)^3}{3} - \frac{2(as)^5}{15} + \ldots\right] \Rightarrow v = a\left[s + \frac{(as)^3}{3} - \frac{2(as)^5}{15} + \ldots\right]$$

The output current I_{OUT} is expressed by variable $v = \Delta U/U_T$ just as before but *v* is now a non-linear function of the input signal current and not simply proportional to the input signal. The hyperbolic tangent is again represented by the Taylor series-expansion.

$$I_{OUT} = I_E \tanh\left(\frac{\Delta U}{2U_T}\right) = I_E \tanh(v) = I_E\left[v - \frac{v^3}{3} + \frac{2v^5}{15} - \ldots\right]$$

The newly derived expression for *v* is substituted into the series and the sequence is terminated after the fifth-order term since the analysis concentrates on the lower harmonics.

$$I_{OUT} \approx I_E\left[a[s + \frac{(as)^3}{3} - \frac{2(as)^5}{15}] - \frac{1}{3}\left(a[s + \frac{(as)^3}{3} - \frac{2(as)^5}{15}]\right)^3 + \frac{2}{15}\left(a[s + \frac{(as)^3}{3} - \frac{2(as)^5}{15}]\right)^5\right]$$

The polynomials raised to third or fifth power consists of several terms containing s^3 or s^5 as well as higher-order terms but the analysis concentrates on the terms until s^5 so terms of higher order are ignored.

$$I_{OUT} = I_E\left[a[s + \frac{(as)^3}{3} - \frac{2(as)^5}{15}] - \frac{a^3[s^3 + a^3 s^5]}{3} + \frac{2a^5 s^5}{15}\right]$$

Assembling terms.

$$I_{OUT} = I_E\left[as - \frac{a^3}{3}(1-a)s^3 - \frac{a^5}{15}(7a-2)s^5\right]$$

For 1 % deviation from linearity the high-order terms must not exceed 0.01 of the first-order term. There are two equations since the series have been terminated after the fifth-order term and the results are approximately equal with a geometrical mean of the two input signals at $s \approx 0.54$, which will be rounded to $s < 0.5$ to simplify matters. Hence, the input current amplitude must be less than half the bias current of the diodes, which seems a quite reasonable requirement since $I_F/2$ is flowing through each diode at equilibrium so the peak input current should not exceed this value.

$$\frac{a^3}{3}(1-a)s^3 < 0.01\,as \Rightarrow s < \sqrt{\frac{0.03}{a^2(1-a)}} \approx 0.62$$

$$\frac{a^5}{15}(7a-2)s^3 < 0.01\,as \Rightarrow s < \sqrt[4]{\frac{0.15}{a^4(7a-2)}} \approx 0.48$$

Put otherwise, the distortion can be expected to increase drastically if the limit is exceeded so this defines a limit *never to be exceeded*, even at the most extreme input condition.

Resultant gain

The small-signal relation gives the nominal signal gain from the input signal U_S to the output signal U_{OUT} assuming linear circuits, which is allowable for an input signal amplitude safely below the limit of $I_S < I_F/2$. The relation from U_S to ΔU is derived from the solution to the transcendent equation. The signal current I_S is assumed given by the input voltage U_S with the proportionality constant R_1 so the relation becomes: $U_S = R_1 I_S$ (thus ignoring the resistance due to R_2 and R_3 in parallel with the dynamic resistance of the diodes D_1 and D_2 since the latter is around 50 Ω for each diode and thus almost shorting the input).

$$v = as \quad \Rightarrow \quad \frac{\Delta U}{2U_T} = \frac{A}{A+1}\frac{I_S}{I_F} \quad \Rightarrow \quad \frac{\Delta U}{2U_T} = \frac{A}{A+1}\frac{U_S}{I_F R_1}$$

The relation from ΔU to U_{OUT} is determined from the idealised equation, which is allowable since the distortion is low using the pre-distortion circuit.

$$U_{OUT} = -R_5 I_{OUT} = -R_5 I_E \frac{\Delta U}{2U_T} = -R_5 I_E \frac{A}{A+1}\frac{U_S}{I_F R_1}$$

The attenuation factor due to A can be written as follows, showing that the value approaches unity for high value of the product $R_2 I_F$ compared to $2U_T$ = 52 mV. This is almost satisfied for I_F = 1 mA and R_2 = 499 Ω since $2U_T/R_2 I_F$ = 0.10 so the gain factor becomes 0.91, which is 9 % below unity.

$$\frac{A}{A+1} = \frac{\frac{R_2 I_F}{2U_T}}{\frac{R_2 I_F}{2U_T}+1} = \frac{1}{1+\frac{2U_T}{R_2 I_F}} \quad \Rightarrow \quad U_{OUT} = -\frac{R_5}{R_1}\frac{I_E}{I_F}\frac{1}{1+\frac{2U_T}{R_2 I_F}} U_S$$

The resultant expression of the gain from input to output is $G = U_{OUT}/U_S$ and is given by the external current I_E and the circuit values:

$$\boxed{U_{OUT} \approx -\frac{R_5}{R_1}\frac{I_E}{I_F} U_S \quad \text{where} \quad U_S < \frac{R_1 I_F}{2}}$$

The gain is 0.91 times with $I_E = I_F$ = 1 mA, $R_1 = R_5$ = 10 kΩ, $R_2 = R_3$ = 499 Ω and R_4 = 11 kΩ where the approximate expression above gives unity. The input resistor has been reduced from 96 kΩ to 10 kΩ thus accounting for the inclusion of the circuit related to the pre-distortion. The current I_E is limited to I_E < 2 mA according to the LM3080 data sheet. The input current is limited to $I_F/2$.

Noise analysis

Thermal noise contributions are the resistors at the input and the noise from the active components within the LM3080; however, there are no data available for the latter so the signal-to-noise ratio will be calculated from the definition of shot noise. The noise voltage from the resistance R_2 within the bandwidth B is given by the usual equation for thermal noise also known as Johnson noise[4]:

$$U_{nR} = \sqrt{4kTBR_2}$$

The resistors in charge are R_2 and R_3 since the contribution from R_1 can safely be ignored. Using an audio bandwidth of B = 20 kHz as noise bandwidth the average noise voltage from each resistor becomes U_{nR} = 410 nV. The total noise contribution from the resistors is calculated by adding power, which results in the usual equation for adding noise voltages:

$$U_{nR} = \sqrt{U_{n1}^2 + U_{n2}^2 + \ldots}$$

4 http://en.wikipedia.org/wiki/Thermal_noise.

Hence the resultant noise voltage of √2 times 410 nV or 580 nV from the resistors in combination. The shot noise from a PN junction (diode or transistor) carrying the current *I* is due to the statistical fluctuations of the finite number of electrons[5]:

$$I_{nT} = \sqrt{2\,qIB}$$

The current can be referred to the input as an equivalent noise voltage source using the relation between I_{OUT} and ΔU with the current through each transistor at $I_E/2$. The LM3080 contains 13 transistors all operating at approximately the same current and connected in series from a signal point of view, so the level is 3.6 times larger since √13 equals 3.6.

$$3.6\,I_{nT} \approx I_{OUT} = I_E \frac{\Delta U}{2U_T} \quad \Rightarrow \quad U_{nT} \approx \Delta U = 2U_T \frac{3.6\,I_{nT}}{I_E} = 7.2\,U_T \sqrt{\frac{qB}{I_E}}$$

For I_E = 1 µA the equivalent noise voltage due to the shot noise becomes 11 µV while the value is 340 nV at 1 mA so the shot noise dominates at low gain. Equal noise contribution is obtained at the current level I_E = 340 µA (for R_2 = 499) so the change is moderate at higher gain levels since the current is limited to I_E = 2 mA maximum.

$$U_{nT} = U_{nR} \quad \Rightarrow \quad 7.2\,U_T \sqrt{\frac{qB}{I_E}} = \sqrt{2}\sqrt{4\,kTBR_2} \quad \Rightarrow \quad \frac{q}{I_E} = \frac{2\cdot 4\,kTR_2}{(7.2\,U_T)^2} \quad \Rightarrow \quad I_E = \frac{q\,(7.2\,U_T)^2}{8\,kTR_2}$$

Using I_F for normalisation (thus not introducing further variables) the noise voltage can be written as follows where U_{nT} = 330 nV for a current of $I_E = I_F$ = 1 mA.

$$U_{nT} = 7.2\,U_T \sqrt{\frac{qB}{I_E}} = 7.2\,U_T \sqrt{\frac{qB}{I_F}} \sqrt{\frac{I_F}{I_E}} \quad \Rightarrow \quad U_{nT} = (330\,\text{nV}) \sqrt{\frac{1\,\text{mA}}{I_E}}$$

At low gain (low I_E) the shot noise dominates so the signal-to-noise ratio becomes function of the external current. For an input signal of ΔU = 10 mV the signal-to-noise ratio is 90 dB at I_E = 1 mA, and the signal-to-noise ratio is reduced by 3 dB for each halving of the current while the ratio is approximately unchanged at higher current levels.

$$\boxed{SNR = 20\cdot\log_{10}\left(\frac{\Delta U}{(330\,\text{nV})} \sqrt{\frac{I_E}{1\,\text{mA}}}\right)}$$

As a conclusion, the design accepts input signals up to ±5 V (R_1 = 10 kΩ and $R_2 = R_3$ = 499 Ω) with an output of ±5 V (R_5 = 10 kΩ and I_E = 1 mA); the third harmonic distortion is 1 % (R_4 = 11 kΩ); the signal-to-noise ratio approximates 90 dB within 20 kHz bandwidth with 3 dB reduction for each halving of the controlling current; and the regulation range is six decades or 120 dB.

References

1 Lennart Råde and Bertil Westergreen “Mathematics Handbook for Science and Engineering”, Studentlitteratur, 5th edition, 2004.

5 http://en.wikipedia.org/wiki/Shot_noise.